Simplification of Differential Algebraic Equations by the Projection Method[1]

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Abstract

Reduction of a differential algebraic equation (DAE) system to an ordinary differential equation system (ODE) is an important step in solving the DAE numerically. When the ODE is obtained, an ODE solution technique can be used to obtain the final solution. In this paper we consider combining index reduction with projection onto the constraint manifold. We show that the reduction benefits from the projection for DAEs of certain form. We demonstrate that one of the applications where DAEs of this form appear is optimization under constraints. We emphasize the importance of optimization problems in physical systems and provide an example application of the projection method to an electric circuit formulated as an optimization problem where Kirchhoff's laws are acting as constraints.

Keywords differential algebraic equations, index reduction, projection method

1. Introduction

The idea of using projection in the treatment of DAEs stems from the projection method introduced by Scott [16] as an alternative to Lagrange's equations for obtaining equations of motion of a constrained mechanical system. Scott proposed to consider constraints as a manifold and to project the equations of unconstrained motion onto the space tangent to the manifold. The idea came from intuitive observations of a mechanical system subject to holonomic constraints and was later justified by Blajer [6]. Blajer [7] and Arczewski and Blajer [1] extended the applications of the projection method to mechanical systems subject to affine linear nonholonomic constraints. From Arczewski and Blajer's [1] derivations it can be seen that the projection method is not an alternative to Lagrange's equations but rather a technique applied to the Lagrange equations in order to simplify them.

To describe the behavior of a constrained mechanical system correctly, the equations of the unconstrained system and the constraints must be combined into one system of equations that takes into account the contributions of the forces induced by the constraints. One of the possible ways to achieve this goal is to introduce the Lagrange multipliers as it is done in Lagrange's equations. The system of DAEs obtained by introducing the Lagrange multipliers is the system to which the projection method is applied. Simply put, the projection method is a method to eliminate the necessity of computing the Lagrange multipliers and to satisfy the constraints in one simple action – projection onto the constraint manifold. At the same time, it is still possible to find the Lagrange multipliers after the solution is obtained [1], if necessary.

From [1] we see that the projection method consists of the index reduction performed by differentiation followed by projection. We want to investigate what we can gain in the case of an arbitrary DAE by adding projection to the index reduction.

Consider a DAE system in a general form given by

$$\mathbf{f}(t, \mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{z}(t)) = 0, \tag{1}$$

where $\mathbf{f} = (f_1, \ldots, f_{n+k})^T$, the superscript T denotes the transpose, t is time, $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))^T$ is called the vector of differential variables, due to the presence of its derivatives $\dot{\mathbf{x}}(t) = (dx_1(t)/dt, \ldots, dx_n(t)/dt)^T$, and $\mathbf{z}(t) = (z_1(t), \ldots z_k(t))^T$ is called the vector of algebraic variables.

System (1) can be reduced to a system of ODEs by differentiating with respect to t. The number of differentiations needed to obtain equations for $\dot{\mathbf{z}}(t)$ is called the *index* of system (1) [2]. The higher the index of a DAE, the more complicated the problem of converting it to an ODE. For example, a mechanical system with holonomic constraints typically is an index-3 DAE system.

In this paper, we consider DAEs of various indices and investigate the properties of the DAEs that make simplification by projection of the reduced system possible. One application where DAEs with these properties are common is optimization under constraints.

We consider optimization problems that result in DAEs and show how addition of projection to the index reduction

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simplifies the equations. As an example, we formulate an optimization problem for an electric circuit consisting of a resistor, an inductor, and a capacitor connected in series and demonstrate the application of the projection method.

2. Preliminaries

In what follows, to make the formulae easily readable, we do not write explicitly what variables a function depends on, unless this information is crucial for understanding the formulae. Moreover, by a linear function or equation we always mean one that is affine linear.

We want to consider the general form of a DAE and investigate how projection can assist us in the reduction of the DAE to an ODE. We want to consider DAEs of different indices. The general form (1) is not suited for this purpose, for it does not reflect the index information. Luckily, most of the problems encountered in practice can be expressed as a combination of more restrictive structures of ODEs coupled with constraints [2]. These are called Hessenberg forms and are given below.

Hessenberg Index-1

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{z}), \tag{2a}$$

$$0 = \mathbf{h}(t, \mathbf{x}, \mathbf{z}), \tag{2b}$$

where $\mathbf{x} = \mathbf{x}(t) = (x_1(t), \dots, x_n(t))^T$ is the vector of differential variables, $\mathbf{f} = (f_1, \dots, f_n)^T$, $\mathbf{z} = \mathbf{z}(t) = (z_1(t), \dots, z_k(t))^T$ is the vector of algebraic variables, $\mathbf{h} = (h_1, \dots, h_k)^T$, and the Jacobian

$$\partial\mathbf{h}/\partial\mathbf{z} \text{ is assumed to be nonsingular for all } t. \tag{3}$$

Equations (2a) are differential, and equations (2b) are algebraic. If the Jacobian is nonsingular, by the implicit function theorem [12], it is possible to solve the algebraic equations (2b) for $\mathbf{z}$ uniquely in a numerical fashion.

We note that the Jacobian in condition (3) may become singular for some particular solution of the system (2). Consequently, the index of the DAE will be higher for this particular solution. Therefore, in what follows, when we speak of index, this is to be understood as the generic index meaning that the corresponding Jacobian is nonsingular for almost all t and for almost all solutions.

Hessenberg Index-2

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{z}), \tag{4a}$$

$$0 = \mathbf{h}(t, \mathbf{x}), \tag{4b}$$

where $\mathbf{x}$, $\mathbf{z}$, $\mathbf{f}$, $\mathbf{h}$ are defined as above for the index-1 form (2), and the product of Jacobians

$$\frac{\partial\mathbf{h}}{\partial\mathbf{x}}\frac{\partial\mathbf{f}}{\partial\mathbf{z}} \text{ is nonsingular for all } t. \tag{5}$$

For example, a mechanical system subject only to nonholonomic constraints is a Hessenberg index-2 system, where $\mathbf{x}$ is the vector of velocities, and $\mathbf{z}$ is the vector of Lagrange multipliers.

Hessenberg Index-3

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{x}, \mathbf{y}, \mathbf{z}), \tag{6a}$$

$$\dot{\mathbf{x}} = \mathbf{g}(t, \mathbf{x}, \mathbf{y}), \tag{6b}$$

$$0 = \mathbf{h}(t, \mathbf{x}), \tag{6c}$$

where $\mathbf{y} = \mathbf{y}(t) = (y_1(t), \dots y_m(t))^T$ and $\mathbf{x} = \mathbf{x}(t) = (x_1(t), \dots x_n(t))^T$ are the vectors of differential variables, $\mathbf{f} = (f_1, \dots, f_m)^T$, $\mathbf{g} = (g, \dots, g_n)^T$, $\mathbf{z} = \mathbf{z}(t) = (z_1(t), \dots, z_k(t))^T$ is the vector of algebraic variables, $\mathbf{h} = (h_1, \dots, h_k)^T$, and the product of the Jacobians

$$\frac{\partial\mathbf{h}}{\partial\mathbf{x}}\frac{\partial\mathbf{g}}{\partial\mathbf{y}}\frac{\partial\mathbf{f}}{\partial\mathbf{z}} \text{ is nonsingular for all } t. \tag{7}$$

There are two different kinds of differential equations, (6a) and (6b), and a set of algebraic equations (6c). An example of a DAE system in Hessenberg index-3 form is a mechanical system subject only to holonomic constraints, whose displacements, velocities and Lagrange multipliers correspond to $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, respectively.

Hessenberg forms of higher index are defined similarly.

Given a Hessenberg index-1 equation of the form (2), by differentiating the algebraic equations (2b) once and replacing $\dot{\mathbf{x}}$ according to equation (2a), equations (2) are reduced to ODEs

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{z}), \tag{8a}$$

$$\dot{\mathbf{z}} = -\left(\frac{\partial\mathbf{h}}{\partial\mathbf{z}}\right)^{-1}\left[\frac{\partial\mathbf{h}}{\partial t} + \frac{\partial\mathbf{h}}{\partial\mathbf{x}}\mathbf{f}(t, \mathbf{x}, \mathbf{z})\right]. \tag{8b}$$

Note that we do not solve the algebraic equations themselves. We use only the derivatives of the algebraic equations in the reduced system (8). In exact computations, we use the initial conditions to satisfy the original algebraic equations. Numerical integration, however, always involves some small errors, which can accumulate into a large error. Consequently, it may happen that a numerical solution of system (8) does not solve the original problem (2) [8]. This phenomenon is known as *constraint drift*. In order to avoid it, the integration procedure must be modified. Therefore, special methods for index-1, index-2, and even Hessenberg index-3 DAEs have been developed [2, 10]. For example, for index-1 DAEs, solving the original algebraic equations by the Newton-Raphson method could be added to every integration step.

In what follows, we take the index-1 form as the target form when performing index reduction, since most numerical integration software can easily handle index-1 DAEs.

3. Reduction of Hessenberg Index-3 DAEs and the Projection Method

As it turns out, not much simplification is possible for Hessenberg index-2 DAEs, and we omit this case here.

Consider now the Hessenberg index-3 system (6). From a geometrical point of view, the algebraic equations (6c) define a time-varying manifold, which we denote by $\mathcal{M}(t)$, in the n-dimensional vector space of $\mathbf{x}$. For $\mathbf{x}$ to stay on $\mathcal{M}(t)$ for every t, $\dot{\mathbf{x}}$ must be tangent to $\mathcal{M}(t)$ for every t. Since $\frac{\partial\mathbf{h}}{\partial\mathbf{x}}\frac{\partial\mathbf{g}}{\partial\mathbf{y}}\frac{\partial\mathbf{f}}{\partial\mathbf{z}}$ is nonsingular,

$$\mathbf{C} = \frac{\partial\mathbf{h}}{\partial\mathbf{x}}$$

is of maximal row-rank, and the rows of the matrix $\mathbf{C}$ span the space normal to $\mathcal{M}(t)$ at $\mathbf{x}(t)$ for every $\mathbf{x}$ and t.

Differentiating the algebraic equations (6c) with respect to time, we get

$$\mathbf{C}\dot{\mathbf{x}} + \frac{\partial \mathbf{h}}{\partial t} = 0. \tag{9}$$

Equation (9) means that the projection of $\dot{\mathbf{x}}$ onto the space normal to $\mathcal{M}(t)$ is determined only by the change of the manifold $\mathcal{M}(t)$ with time. If the manifold is time-invariant, i.e. the algebraic equations are time-invariant, then $\dot{\mathbf{x}}$ is tangent to $\mathcal{M}(t)$ for any t.

To describe the projection of $\dot{\mathbf{x}}$ onto the tangent space, we choose a matrix $\mathbf{D}$ such that $\mathbf{D}$ is of maximal column-rank and

$$\mathbf{CD} = 0, \tag{10}$$

i.e. the columns of $\mathbf{D}$ span the space tangent to $\mathcal{M}(t)$ at $\mathbf{x}(t)$ for every $\mathbf{x}$ and t. Therefore, for any t we have obtained two subspaces of $\mathbb{R}^n$, spanned by the rows of $\mathbf{C}$ and by the columns of $\mathbf{D}$, that are orthogonal complements of each other. By the theorem on orthogonal decomposition [11], any element in $\mathbb{R}^n$ can be represented as a sum of projections onto these subspaces. Consequently, for any t,

$$\dot{\mathbf{x}} = \mathbf{D}\mathbf{u} + \mathbf{C}^T\mathbf{v}, \tag{11}$$

for certain vectors $\mathbf{u} \in \mathbb{R}^{n-k}$ and $\mathbf{v} \in \mathbb{R}^k$. Note that both $\mathbf{C}$ and $\mathbf{D}$ depend on $\mathbf{x}$ and t only, but not on $\mathbf{y}$ or $\mathbf{z}$.

Substituting representation (11) into equation (9), we find the tangential component of $\dot{\mathbf{x}}$ as

$$\mathbf{v} = -\left(\mathbf{C}\mathbf{C}^T\right)^{-1}\frac{\partial \mathbf{h}}{\partial t}, \tag{12}$$

vanishing identically if $\mathbf{h}$ does not depend on t explicitly.

We start the reduction of system (6) to an index-1 system by differentiating the algebraic equations (6c), as in (9), and then use equation (6b) to obtain:

$$\mathbf{C}\mathbf{g} + \frac{\partial \mathbf{h}}{\partial t} = 0. \tag{13}$$

We differentiate equation (13) once more:

$$\mathbf{C}\left[\frac{\partial \mathbf{g}}{\partial t} + \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{g}}{\partial \mathbf{y}}\dot{\mathbf{y}}\right] + \dot{\mathbf{C}}\mathbf{g} + \frac{\partial^2 \mathbf{h}}{\partial t^2} + \frac{\partial^2 \mathbf{h}}{\partial \mathbf{x}\partial t}\dot{\mathbf{x}} = 0. \tag{14}$$

Finally, using equations (6a) and (6b) in equation (14), we can re-write equations (6) as:

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{x}, \mathbf{y}, \mathbf{z}), \tag{15a}$$

$$\dot{\mathbf{x}} = \mathbf{g}(t, \mathbf{x}, \mathbf{y}), \tag{15b}$$

$$0 = \dot{\mathbf{C}}\mathbf{g} + \mathbf{C}\left[\frac{\partial \mathbf{g}}{\partial t} + \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{g} + \frac{\partial \mathbf{g}}{\partial \mathbf{y}}\mathbf{f}\right] + \frac{\partial^2 \mathbf{h}}{\partial t^2} + \frac{\partial^2 \mathbf{h}}{\partial \mathbf{x}\partial t}\mathbf{g}. \tag{15c}$$

The system (15) is an index-1 system of DAEs. In order to see this, we notice that $\mathbf{z}$ appears in the equations (15c) only inside $\mathbf{f}$, and the Jacobian in condition (3) for the equations (15c) is equal to the product of Jacobians in condition (7). Then condition (7) yields that the algebraic equations (15c) are solvable for $\mathbf{z}$.

We now want to apply the projection method to simplify the system (6) and compare the result with system (15). We substitute the representation (11) of $\dot{\mathbf{x}}$ into equation (6b):

$$\mathbf{D}\mathbf{u} + \mathbf{C}^T\mathbf{v} = \mathbf{g}(t, \mathbf{x}, \mathbf{y}). \tag{16}$$

Instead of differentiating equation (13), we differentiate equation (16) and obtain

$$\dot{\mathbf{D}}\mathbf{u} + \mathbf{D}\dot{\mathbf{u}} + \dot{\mathbf{C}}^T\mathbf{v} + \mathbf{C}^T\dot{\mathbf{v}} = \frac{\partial \mathbf{g}}{\partial t} + \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{g} + \frac{\partial \mathbf{g}}{\partial \mathbf{y}}\mathbf{f}, \tag{17}$$

where $\mathbf{v}$ is given by equation (12). Projecting equation (17) onto the normal space by multiplying it by $\mathbf{C}$ on the left, we obtain equations for $\mathbf{z}$:

$$\mathbf{C}\dot{\mathbf{D}}\mathbf{u} + \mathbf{C}\dot{\mathbf{C}}^T\mathbf{v} + \mathbf{C}\mathbf{C}^T\dot{\mathbf{v}} = \mathbf{C}\frac{\partial \mathbf{g}}{\partial t} + \mathbf{C}\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{g} + \mathbf{C}\frac{\partial \mathbf{g}}{\partial \mathbf{y}}\mathbf{f}, \tag{18}$$

where we used relation (10) between $\mathbf{C}$ and $\mathbf{D}$. Due to condition (7), equation (18) is solvable for $\mathbf{z}$.

Projecting (17) onto the tangential space by multiplying it by $\mathbf{D}^T$ on the left, we obtain equations for $\dot{\mathbf{u}}$:

$$\mathbf{D}^T\dot{\mathbf{D}}\mathbf{u} + \mathbf{D}^T\mathbf{D}\dot{\mathbf{u}} + \mathbf{D}^T\dot{\mathbf{C}}^T\mathbf{v} = \mathbf{D}^T\frac{\partial \mathbf{g}}{\partial t} + \mathbf{D}^T\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{g} + \mathbf{D}^T\frac{\partial \mathbf{g}}{\partial \mathbf{y}}\mathbf{f}, \tag{19}$$

where we used relation (10) between $\mathbf{C}$ and $\mathbf{D}$ again. Since $\mathbf{D}$ is of maximal column-rank, from (19) we can explicitly solve for $\dot{\mathbf{u}}$ in terms of t, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{u}$. Note that from equation (12), it follows that $\mathbf{v}$ is a known function of t and $\mathbf{x}$. However, to complete the change of variables from $\mathbf{y}$ to $\mathbf{u}$, we also need a representation of $\mathbf{y}$ in terms of t, $\mathbf{x}$ and $\mathbf{u}$. To find the relation between $\mathbf{y}$ and $\mathbf{u}$, we use (16).

In order to determine whether the equations (16) are solvable for $\mathbf{y}$, we take a closer look at condition (7). From condition (7), it follows that rank $\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\frac{\partial \mathbf{g}}{\partial \mathbf{y}}\frac{\partial \mathbf{f}}{\partial \mathbf{z}} = k$, and, hence,

$$\operatorname{rank}\frac{\partial \mathbf{h}}{\partial \mathbf{x}} \geq k, \tag{20}$$

$$\operatorname{rank}\frac{\partial \mathbf{g}}{\partial \mathbf{y}} \geq k, \tag{21}$$

$$\operatorname{rank}\frac{\partial \mathbf{f}}{\partial \mathbf{z}} \geq k. \tag{22}$$

Moreover, we can conclude that

$$n \geq k \text{ and } m \geq k. \tag{23}$$

To verify relations (23), assume otherwise: Let $n < k$. Then the rank of $\partial\mathbf{h}/\partial\mathbf{x}$ cannot be higher than n, and, therefore, not higher than k, either, which contradicts relation (20). Similarly, if $m < k$, the rank of $\partial\mathbf{f}/\partial\mathbf{z}$ cannot be higher than m, and, therefore, not higher than k, either, which contradicts relation (22). Thus, relations (23) hold.

Let the rank of $\partial\mathbf{g}/\partial\mathbf{y}$ be equal to j. In what follows, for any vector or matrix, the superscript a denotes the first j rows, e.g. $\mathbf{y}^a = (y_1, \ldots, y_j)^T$, and the superscript d denotes the last rows starting from $j+1$, e.g. $\mathbf{y}^d = (y_{j+1}, \ldots, y_m)^T$. For simplicity, we re-order the components of $\mathbf{y}$ so that rank $\frac{\partial \mathbf{g}}{\partial \mathbf{y}}$ = rank $\frac{\partial \mathbf{g}}{\partial \mathbf{y}^a} = j$. Then equations (16) can be considered as algebraic equations with respect to the $\mathbf{y}^a$. The variables $\mathbf{y}^d$ are differential variables whose differential equations are given by

$$\dot{\mathbf{y}}^d = \mathbf{f}^d. \tag{24}$$

The projected index-1 DAE obtained from (11), (16), (18), (19), and (24) is given by

$$\dot{\mathbf{x}} = \mathbf{D}\mathbf{u} + \mathbf{C}^T\mathbf{v}, \tag{25a}$$

$$\dot{\mathbf{u}} = \left(\mathbf{D}^T\mathbf{D}\right)^{-1}\mathbf{D}^T\mathbf{w}, \tag{25b}$$

$$\dot{\mathbf{y}}^d = \mathbf{f}^d, \tag{25c}$$

$$0 = \mathbf{D}\mathbf{u} + \mathbf{C}^T\mathbf{v} - \mathbf{g}, \tag{25d}$$

$$0 = \mathbf{C}\left[\mathbf{w} - \mathbf{C}^T\dot{\mathbf{v}}\right], \tag{25e}$$

where $\mathbf{v}$ is given by equations (12) and

$$\mathbf{w} = \left[\frac{\partial\mathbf{g}}{\partial t} + \frac{\partial\mathbf{g}}{\partial\mathbf{x}}\mathbf{g} + \frac{\partial\mathbf{g}}{\partial\mathbf{y}^a}\mathbf{f}^a + \frac{\partial\mathbf{g}}{\partial\mathbf{y}^d}\mathbf{f}^d - \dot{\mathbf{D}}\mathbf{u} - \dot{\mathbf{C}}^T\mathbf{v}\right]. \tag{26}$$

System (25) consists of $n + (n-k) + (m-j)$ differential equations (25a)-(25c), with respect to $n+(n-k)+(m-j)$ differential variables, $\mathbf{x}$, $\mathbf{u}$, and $\mathbf{y}^d$, respectively, and $n+k$ algebraic equations (25d) and (25e) with respect to $j+k$ algebraic variables, $\mathbf{y}^a$ and $\mathbf{z}$, respectively.

Note that equation (25e) is equation (15c) written in terms of $\mathbf{v}$, $\dot{\mathbf{C}}$, and $\dot{\mathbf{D}}$ and not in terms of derivatives of $\mathbf{h}$ explicitly.

The number of differential equations in system (15) obtained by index reduction is $n+m$, whereas the number of differential equations in system (25) obtained by the projection method is $n+m+(n-k-j)$. If $n > k+j$, an application of the projection method is not more beneficial than an application of the index reduction alone. Otherwise, the projection method performs not only index reduction but also decreases the number of differential variables. In addition, there is a number of special cases that allow to simplify system (25) further.

Special case 1. *Some of the algebraic constraints* (6c) *are linear with respect to* $\mathbf{x}$.

We are given a Hessenberg index-3 system of the form

$$\dot{\mathbf{y}} = \mathbf{f}(t,\mathbf{x},\mathbf{y},\mathbf{z}), \tag{27a}$$

$$\dot{\mathbf{x}} = \mathbf{g}(t,\mathbf{x},\mathbf{y}), \tag{27b}$$

$$0 = \mathbf{C}_1(t)\mathbf{x} + \alpha(t), \tag{27c}$$

$$0 = \mathbf{h}_2(t,\mathbf{x}), \tag{27d}$$

where $\mathbf{C}_1$ is an $l \times n$ matrix of maximal row rank, such that $l \le n$, and $\mathbf{h}_2$ is a $(k-l)$-dimensional vector function which is nonlinear with respect to $\mathbf{x}$.

First, we consider equations (27b) and (27c). Similarly to the projection method, we choose a matrix $\mathbf{D}_1$ of maximal column-rank and such that

$$\mathbf{C}_1\mathbf{D}_1 = 0. \tag{28}$$

Then, since for every t the rows of $\mathbf{C}_1$ span a subspace in $\mathbb{R}^n$ and the columns of $\mathbf{D}_1$ span the orthogonal complement to that subspace, any $\mathbf{x} \in \mathbb{R}^n$ can be represented as

$$\mathbf{x} = \mathbf{D}_1\chi + \mathbf{C}_1^T\psi, \tag{29}$$

where $\chi = (\chi_1,\ldots,\chi_{n-l})^T$ and $\psi = (\psi_1,\ldots,\psi_l)^T$. Substituting decomposition (29) into equation (27c) and making use of relation (28), we find

$$\psi = -\left(\mathbf{C}_1\mathbf{C}_1^T\right)^{-1}\alpha. \tag{30}$$

From equation (29) we find $\dot{\mathbf{x}}$:

$$\dot{\mathbf{x}} = \dot{\mathbf{D}}_1\chi + \mathbf{D}_1\dot{\chi} + \dot{\mathbf{C}}_1^T\psi + \mathbf{C}_1^T\dot{\psi}.$$

We substitute this equation into equation (27b) and multiply the result by $\mathbf{D}_1^T$ on the left, obtaining a differential equation for $\dot{\chi}$:

$$\dot{\chi} = \left(\mathbf{D}_1^T\mathbf{D}_1\right)^{-1}\mathbf{D}_1^T\left[\mathbf{g}(t,\mathbf{D}_1\chi + \mathbf{C}_1^T\psi,\mathbf{y}) - \dot{\mathbf{D}}_1\chi - \dot{\mathbf{C}}_1^T\psi\right]. \tag{31}$$

On the other hand, we can apply the projection method and derive equations (25), where

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_1 \\ \frac{\partial\mathbf{h}_2}{\partial\mathbf{x}} \end{pmatrix}.$$

We see that equations (25a) for $\dot{\mathbf{x}}$ can be replaced with equations (31). Thus, we reduce the DAEs (15) to the following DAEs:

$$\dot{\chi} = \left(\mathbf{D}_1^T\mathbf{D}_1\right)^{-1}\mathbf{D}_1^T\left[\mathbf{g} - \dot{\mathbf{D}}_1\chi - \dot{\mathbf{C}}_1^T\psi\right], \tag{32a}$$

$$\dot{\mathbf{u}} = \left(\mathbf{D}^T\mathbf{D}\right)^{-1}\mathbf{D}^T w, \tag{32b}$$

$$\dot{\mathbf{y}}^d = \mathbf{f}^d, \tag{32c}$$

$$0 = \mathbf{D}\mathbf{u} + \mathbf{C}^T\mathbf{v} - \mathbf{g}, \tag{32d}$$

$$0 = \mathbf{C}\left[w - \mathbf{C}^T\dot{\mathbf{v}}\right], \tag{32e}$$

where w is as in (26), we substituted $\mathbf{x} = \mathbf{D}_1\chi + \mathbf{C}_1^T\psi$, ψ is given by equation (30), and $\mathbf{v}$ is given by equation (12).

System (32) has $(n-l)+(n-k)+(m-j)$ differential variables χ, $\mathbf{u}$, $\mathbf{y}^d$ and $j+k$ algebraic variables $\mathbf{y}^a$, $\mathbf{z}$. □

Special case 2. *Equation* (25d) *can be solved symbolically for* $\mathbf{y}^a$*:* $\mathbf{y}^a = \widetilde{\mathbf{y}}^a(t,\mathbf{x},\mathbf{u},\mathbf{y}^d)$.

Then we can consider the system of equations (25a), (25b), (25c), and (25e) with $\mathbf{y}^a = \widetilde{\mathbf{y}}^a(t,\mathbf{x},\mathbf{u},\mathbf{y}^d)$ substituted, which we solve for $\mathbf{x}$, $\mathbf{u}$, $\mathbf{y}^d$, and $\mathbf{z}$. Consequently, j algebraic equations are removed from the system.

One possible situation where we can express $\mathbf{y}^a$ symbolically is the case where $\mathbf{g}$ is linear with respect to $\mathbf{y}^a$:

$$\mathbf{g}(t,\mathbf{x},\mathbf{y}^a,\mathbf{y}^d) = \mathbf{A}(t,\mathbf{x},\mathbf{y}^d)\mathbf{y}^a + \gamma(t,\mathbf{x},\mathbf{y}^d), \tag{33}$$

and $\mathbf{A}(t,\mathbf{x},\mathbf{y}^d)$ is of maximal column-rank. □

Special case 3. *Equation* (25e) *can be solved symbolically for* $\mathbf{z}$*:*$\mathbf{z} = \widetilde{\mathbf{z}}(t,\mathbf{x},\mathbf{u},\mathbf{y}^a,\mathbf{y}^d)$.

Then we can consider the system of equations (25a), (25b), (25c), and (25d) where $\mathbf{z} = \widetilde{\mathbf{z}}(t,\mathbf{x},\mathbf{u},\mathbf{y}^a,\mathbf{y}^d)$ was substituted, which we solve for $\mathbf{x}$, $\mathbf{u}$, $\mathbf{y}^a$, and $\mathbf{y}^d$, and k algebraic equations are removed from the system.

One possible situation where we can express $\mathbf{z}$ symbolically is the case where $\mathbf{f}$ is linear with respect to $\mathbf{z}$:

$$\mathbf{f}(t,\mathbf{x},\mathbf{y}^a,\mathbf{y}^d,\mathbf{z}) = \varphi(t,\mathbf{x},\mathbf{y}^a,\mathbf{y}^d) + \mathbf{B}(t,\mathbf{x},\mathbf{y}^a,\mathbf{y}^d)\mathbf{z}. \tag{34}$$

In order to satisfy condition (7), $\mathbf{B}$ must be such that $\frac{\partial\mathbf{h}}{\partial\mathbf{x}}\frac{\partial\mathbf{g}}{\partial\mathbf{y}}\mathbf{B}$ is nonsingular for all t. □

When we consider Special cases 2 and 3 together, we can obtain an ODE system with respect to $n+(n-k)+(m-j)$ variables $\mathbf{x}$, $\mathbf{u}$, and $\mathbf{y}^d$. Let us consider what we obtain in the case where $\mathbf{f}$ and $\mathbf{g}$ are given by equations (34) and (33), respectively:

$$\dot{\mathbf{x}} = \mathbf{D}\mathbf{u} + \mathbf{C}^T\mathbf{v}, \tag{35a}$$

$$\dot{\mathbf{u}} = \left(\mathbf{D}^T\mathbf{D}\right)^{-1}\mathbf{D}^T\left[\mathbf{P} + \mathbf{Q}\widetilde{\mathbf{z}} - \dot{\mathbf{D}}\mathbf{u} - \dot{\mathbf{C}}^T\mathbf{v}\right], \tag{35b}$$

$$\dot{\mathbf{y}}^d = \widetilde{\varphi}_d + \widetilde{\mathbf{B}}_d\widetilde{\mathbf{z}}, \tag{35c}$$

where

$$\mathbf{P} = \frac{\partial \mathbf{A}}{\partial t}\widetilde{\mathbf{y}}^a + \frac{\partial \gamma}{\partial t} + \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}}\widetilde{\mathbf{y}}^a) + \frac{\partial \gamma}{\partial \mathbf{x}}\right)(\mathbf{A}\widetilde{\mathbf{y}}^a + \gamma) + \mathbf{A}\widetilde{\varphi}^a + \left(\frac{\partial \mathbf{A}}{\partial \mathbf{y}^d}\widetilde{\mathbf{y}}^a + \frac{\partial \gamma}{\partial \mathbf{y}^d}\right)\widetilde{\varphi}^d, \tag{36a}$$

$$\mathbf{Q} = \mathbf{A}\widetilde{\mathbf{B}}_a + \left(\frac{\partial \mathbf{A}}{\partial \mathbf{y}^d}\widetilde{\mathbf{y}}^a + \frac{\partial \gamma}{\partial \mathbf{y}^d}\right)\widetilde{\mathbf{B}}^d, \tag{36b}$$

$$\widetilde{\mathbf{B}} = \mathbf{B}(t, \mathbf{x}, \widetilde{\mathbf{y}}^a(t, \mathbf{x}, \mathbf{u}, \mathbf{y}^d), \mathbf{y}^d), \tag{36c}$$

$$\widetilde{\varphi} = \varphi(t, \mathbf{x}, \widetilde{\mathbf{y}}^a(t, \mathbf{x}, \mathbf{u}, \mathbf{y}^d), \mathbf{y}^d), \tag{36d}$$

$$\widetilde{\mathbf{y}}^a = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\left[\mathbf{D}\mathbf{u} + \mathbf{C}^T\mathbf{v} - \gamma\right], \tag{36e}$$

$$\widetilde{\mathbf{z}} = (\mathbf{C}\mathbf{Q})^{-1}\mathbf{C}\left[\dot{\mathbf{D}}\mathbf{u} + \dot{\mathbf{C}}^T\mathbf{v} + \mathbf{C}^T\dot{\mathbf{v}} - \mathbf{P}\right]. \tag{36f}$$

When system (35) is solved, $\mathbf{y}^a$ is given by $\widetilde{\mathbf{y}}^a$ from equation (36e), and $\mathbf{z}$ is given by $\widetilde{\mathbf{z}}$ from equation (36f). Equations (35) have the simplest form we can get by applying projection to the reduced system (15).

REMARK 1. *The original projection method [16, 6, 7, 1] was developed for mechanical systems under holonomic constraints:*

$$\mathbf{M}\ddot{\mathbf{x}} = \varphi(t, \mathbf{x}, \dot{\mathbf{x}}) + \mathbf{C}^T\mathbf{z} \tag{37a}$$

$$0 = \mathbf{h}(t, \mathbf{x}), \tag{37b}$$

where $\mathbf{x}$ *denotes the coordinates,* $\mathbf{z}$ *denote the Lagrange multipliers,* $\mathbf{M}$ *is a symmetric positive-definite generalized mass matrix,* φ *is independent of* $\mathbf{z}$ *and represents forces acting on the system. Denoting the velocities by* $\mathbf{y}$*, we can re-write the system* (37) *in the form described by Special cases 2 and 3 with* $\mathbf{y} = \mathbf{y}^a$, $\mathbf{g} = \mathbf{y}$*, and* $\mathbf{f}$ *linear with respect to* $\mathbf{z}$*. However, in this case the computations simplify if we first multiply both sides of equation* (17) *by* $\mathbf{M}$ *on the left and then project the result onto the tangential space, and use equation* (17) *as is when projecting onto the normal space. Equations* (37) *are simplified to:*

$$\dot{\mathbf{x}} = \mathbf{D}\mathbf{u} + \mathbf{C}^T\mathbf{v},$$

$$\dot{\mathbf{u}} = \left(\mathbf{D}^T\mathbf{M}\,\mathbf{D}\right)^{-1}\mathbf{D}^T\left[\widetilde{\varphi} - \mathbf{M}\left(\dot{\mathbf{D}}\mathbf{u} + \dot{\mathbf{C}}^T\mathbf{v} + \mathbf{C}^T\dot{\mathbf{v}}\right)\right],$$

where we used $\mathbf{C}\mathbf{D} = 0$ *and* $\widetilde{\varphi} = \varphi(t, \mathbf{x}, \widetilde{\mathbf{y}})$*. Equations* (36e) *and* (36f)*, respectively, become:*

$$\widetilde{\mathbf{y}} = \mathbf{D}\mathbf{u} + \mathbf{C}^T\mathbf{v},$$

$$\widetilde{\mathbf{z}} = \left(\mathbf{C}\,\mathbf{M}^{-1}\mathbf{C}^T\right)^{-1}\mathbf{C}\left[\dot{\mathbf{D}}\mathbf{u} + \dot{\mathbf{C}}^T\mathbf{v} + \mathbf{C}^T\,\dot{\mathbf{v}} - \mathbf{M}^{-1}\widetilde{\varphi}\right].$$

The last equation is not needed to obtain the motion of the mechanical system but can be used to find the constraint forces, since the vector $\mathbf{C}^T\mathbf{z}$ *represents the generalized forces induced by the constraints.*

Lagrange multipliers $\mathbf{z}$ appear in the equations for a constrained mechanical system (37) as a result of application of the *Principle of Least Action* [4, 13]. For a constrained mechanical system, this principle results in a problem of optimization under constraints. We can show that any optimization under constraints with an objective function similar to the mechanical action gives a DAE system that can be simplified to an ODE system by the projection method. We demonstrate this in Section 6.

4. The Projection Method and Higher Hessenberg Index DAEs

In this section we generalize the approach from the previous section to DAEs of arbitrary Hessenberg index.

We consider a Hessenberg index-$(r+2)$ DAE

$$\dot{\mathbf{y}}_r = \mathbf{f}_r(t, \mathbf{x}, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r, \mathbf{z}), \tag{38a}$$

$$\vdots$$

$$\dot{\mathbf{y}}_2 = \mathbf{f}_2(t, \mathbf{x}, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3), \tag{38b}$$

$$\dot{\mathbf{y}}_1 = \mathbf{f}_1(t, \mathbf{x}, \mathbf{y}_1, \mathbf{y}_2), \tag{38c}$$

$$\dot{\mathbf{x}} = \mathbf{g}(t, \mathbf{x}, \mathbf{y}_1), \tag{38d}$$

$$0 = \mathbf{h}(t, \mathbf{x}), \tag{38e}$$

where

$$\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\frac{\partial \mathbf{g}}{\partial \mathbf{y}_1}\frac{\partial \mathbf{f}_1}{\partial \mathbf{y}_2}\cdots\frac{\partial \mathbf{f}_{r-1}}{\partial \mathbf{y}_r}\frac{\partial \mathbf{f}_r}{\partial \mathbf{z}} \text{ is nonsingular for all } t. \tag{39}$$

The vectors $\mathbf{y}_1$, $\mathbf{y}_2, \ldots,$ $\mathbf{y}_r$ are of size $m_1, m_2, \ldots, m_r$, respectively. Due to condition (39), the ranks of the Jacobians $\frac{\partial \mathbf{g}}{\partial \mathbf{y}_1}, \frac{\partial \mathbf{f}_1}{\partial \mathbf{y}_2}, \ldots, \frac{\partial \mathbf{f}_{r-1}}{\partial \mathbf{y}_r}$ and their products forming subproducts in condition (39) are at least k at any t. We reorder the components of the vectors $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r$ so that the first $j_1, j_2, \ldots, j_r$ components, respectively, denoted by superscript a, are such that:

$$\text{rank}\,\frac{\partial \mathbf{g}}{\partial \mathbf{y}_1} = \text{rank}\,\frac{\partial \mathbf{g}}{\partial \mathbf{y}_1^a} = j_1, \tag{40a}$$

$$\text{rank}\,\frac{\partial \mathbf{g}}{\partial \mathbf{y}_1}\frac{\partial \mathbf{f}_1}{\partial \mathbf{y}_2} = \text{rank}\,\frac{\partial \mathbf{g}}{\partial \mathbf{y}_1}\frac{\partial \mathbf{f}_1}{\partial \mathbf{y}_2^a} = j_2, \tag{40b}$$

$$\vdots$$

$$\text{rank}\,\frac{\partial \mathbf{g}}{\partial \mathbf{y}_1}\frac{\partial \mathbf{f}_1}{\partial \mathbf{y}_2}\cdots\frac{\partial \mathbf{f}_{r-1}}{\partial \mathbf{y}_r} = \text{rank}\,\frac{\partial \mathbf{g}}{\partial \mathbf{y}_1}\frac{\partial \mathbf{f}_1}{\partial \mathbf{y}_2}\cdots\frac{\partial \mathbf{f}_{r-1}}{\partial \mathbf{y}_r^a} = j_r. \tag{40c}$$

As in the previous cases the projection method leads to the representation

$$\dot{\mathbf{x}} = \mathbf{D}\mathbf{u}_1 + \mathbf{C}^T\mathbf{v}, \tag{41}$$

where $\mathbf{C} = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}$ is of maximal row-rank, $\mathbf{D}$ is of maximal column-rank and satisfies (10), $\mathbf{v} = -\left(\mathbf{C}\mathbf{C}^T\right)^{-1}\frac{\partial \mathbf{h}}{\partial t}$, and $\mathbf{u}_1$ is an $(n-k)$-dimensional vector. Then we introduce $(n-k)$-dimensional vectors $\mathbf{u}_2, \ldots, \mathbf{u}_r$ such that

$$\dot{\mathbf{u}}_1 = \mathbf{u}_2, \tag{42a}$$

$$\dot{\mathbf{u}}_2 = \mathbf{u}_3, \tag{42b}$$

$$\vdots$$

$$\dot{\mathbf{u}}_{r-1} = \mathbf{u}_r, \tag{42c}$$

and we want to re-write equations (38) in terms of the projection of $\dot{\mathbf{x}}$ onto the tangential space, $\mathbf{u}_1$, and its derivatives. We substitute (41) into equation (38d), obtaining

$$\mathbf{D}\mathbf{u}_1 + \mathbf{C}^T\mathbf{v} = \mathbf{g}(t, \mathbf{x}, \mathbf{y}_1). \tag{43}$$

From (40a), it follows that (43) can be solved for $\mathbf{y}_1^a$.

Differentiating (43) with respect to time, we obtain

$$\dot{\mathbf{D}}\mathbf{u}_1+\mathbf{D}\mathbf{u}_2+\dot{\mathbf{C}}^T\mathbf{v}+\mathbf{C}^T\dot{\mathbf{v}}=\frac{\partial\mathbf{g}}{\partial t}+\frac{\partial\mathbf{g}}{\partial\mathbf{x}}\mathbf{g}+\frac{\partial\mathbf{g}}{\partial\mathbf{y}_1}\mathbf{f}_1, \quad (44)$$

where we made use of (38d), (38c), and (42a). From condition (40b), it follows that (44) can be solved for $\mathbf{y}_2^a$.

Differentiating equation (43) with respect to time a second time, i.e. differentiating equation (44), and replacing derivatives of variables that occur in the left-hand side with their representation according to equations (41) and (42) and derivatives that occur in the right-hand side according to equations (38), we obtain algebraic equations that can be solved for $\mathbf{y}_3^a$. We continue this process until we differentiate equation (43) r times. After each differentiation we make use of equations (41) and (42) in the left-hand side and of equations (38) on the right-hand side. The i-th differentiation gives us an algebraic equation, which due to conditions (40) can be solved for $\mathbf{y}_{i+1}^a$, for $i=1,\ldots,r-1$.

The r-th differentiation gives us an equation, from which we derive a differential equation for $\dot{\mathbf{u}}_r$ and an algebraic equation for $\mathbf{z}$. Let us consider the result of r-th differentiation. We are interested only in the terms that define equations for $\dot{\mathbf{u}}_r$ and $\mathbf{z}$, and, therefore, for simplicity, we represent the result of r-th differentiation as follows:

$$\begin{aligned}&\mathbf{D}\dot{\mathbf{u}}_r+\varphi_1\left(t,\mathbf{x},\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_r\right) && (45)\\ &=\frac{\partial\mathbf{g}}{\partial\mathbf{y}_1}\frac{\partial\mathbf{f}_1}{\partial\mathbf{y}_2}\frac{\partial\mathbf{f}_2}{\partial\mathbf{y}_3}\cdots\frac{\partial\mathbf{f}_{r-1}}{\partial\mathbf{y}_r}\mathbf{f}_r+\varphi_2\left(t,\mathbf{x},\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_r\right),\end{aligned}$$

where functions φ_1 and φ_2 represent the remaining terms, in which we are not interested.

If we project equation (45) onto the tangential space of the constraint manifold by multiplying both sides of the equation by $\mathbf{D}^T$ on the left, we obtain an equation for $\dot{\mathbf{u}}_r$.

$$\dot{\mathbf{u}}_r=\left(\mathbf{D}^T\mathbf{D}\right)^{-1}\mathbf{D}^T\mathbf{e}, \quad (46)$$

where

$$\mathbf{e}=\left[\frac{\partial\mathbf{g}}{\partial\mathbf{y}_1}\frac{\partial\mathbf{f}_1}{\partial\mathbf{y}_2}\frac{\partial\mathbf{f}_2}{\partial\mathbf{y}_3}\cdots\frac{\partial\mathbf{f}_{r-1}}{\partial\mathbf{y}_r}\mathbf{f}_r+\varphi_2-\varphi_1\right]. \quad (47)$$

If we project equation (45) onto the normal space of the constraint manifold by multiplying both sides of the equation by $\mathbf{C}$ on the left, we obtain an equation for $\mathbf{z}$, which is solvable due to condition (39) and is independent of $\dot{\mathbf{u}}_r$ due to relation (10).

$$0=\left(\mathbf{C}\mathbf{C}^T\right)^{-1}\mathbf{C}\,\mathbf{e} \quad (48)$$

The DAE system simplified by the projection method is then given by equations (41), (42), (43), (44), (46), (48), algebraic equations for $\mathbf{y}_3^a,\ldots,\mathbf{y}_r^a$ and differential equations for $\mathbf{y}_1^d,\ldots,\mathbf{y}_r^d$, where we represent each $\mathbf{y}_i$ in terms of $\mathbf{y}_i^a$ and $\mathbf{y}_i^d$, for $i=1,\ldots,r$.

$$\begin{aligned}
\dot{\mathbf{x}}&=\mathbf{D}\mathbf{u}_1+\mathbf{C}^T\mathbf{v}, && (49a)\\
\dot{\mathbf{u}}_1&=\mathbf{u}_2, && (49b)\\
\dot{\mathbf{u}}_2&=\mathbf{u}_3, && (49c)\\
&\vdots\\
\dot{\mathbf{u}}_{r-1}&=\mathbf{u}_r, && (49d)\\
\dot{\mathbf{u}}_r&=\left(\mathbf{D}^T\mathbf{D}\right)^{-1}\mathbf{D}^T\mathbf{e}, && (49e)\\
\dot{\mathbf{y}}_1^d&=\mathbf{f}_1^d, && (49f)\\
\dot{\mathbf{y}}_2^d&=\mathbf{f}_2^d, && (49g)\\
&\vdots\\
\dot{\mathbf{y}}_r^d&=\mathbf{f}_r^d, && (49h)\\
0&=\mathbf{D}\mathbf{u}_1+\mathbf{C}^T\mathbf{v}-\mathbf{g}, && (49i)\\
0&=\dot{\mathbf{D}}\mathbf{u}_1+\mathbf{D}\mathbf{u}_2+\dot{\mathbf{C}}^T\mathbf{v}+\mathbf{C}^T\dot{\mathbf{v}}-\frac{\partial\mathbf{g}}{\partial t}-\frac{\partial\mathbf{g}}{\partial\mathbf{x}}\mathbf{g}-\frac{\partial\mathbf{g}}{\partial\mathbf{y}_1}\mathbf{f}_1, && (49j)\\
0&=\frac{\mathrm{d}^3}{\mathrm{d}t^3}\left(\mathbf{D}\mathbf{u}_1+\mathbf{C}^T\mathbf{v}\right)\Big|_{(41),(42)}-\frac{\mathrm{d}^3}{\mathrm{d}t^3}\left(\mathbf{g}\right)\Big|_{(38)}, && (49k)\\
&\vdots\\
0&=\frac{\mathrm{d}^{r-1}}{\mathrm{d}t^{r-1}}\left(\mathbf{D}\mathbf{u}_1+\mathbf{C}^T\mathbf{v}\right)\Big|_{(41),(42)}-\frac{\mathrm{d}^{r-1}}{\mathrm{d}t^{r-1}}\left(\mathbf{g}\right)\Big|_{(38)}, && (49l)\\
0&=\left(\mathbf{C}\mathbf{C}^T\right)^{-1}\mathbf{C}\,\mathbf{e}. && (49m)
\end{aligned}$$

System (49) consists of differential equations (49a)-(49h) w.r.t. $n+r(n-k)+(m_1-j_1)+\ldots+(m_r-j_r)$ variables $\mathbf{x},\mathbf{u}_1,\ldots,\mathbf{u}_r,\mathbf{y}_1^d,\ldots,\mathbf{y}_r^d$ and of algebraic equations (49i)-(49m) w.r.t. $j_1+j_2+\ldots+j_r+k$ variables $\mathbf{y}_1^a,\ldots,\mathbf{y}_r^a,\mathbf{z}$. Index reduction alone yields an index-1 DAE w.r.t. $n+m_1+\ldots+m_r$ differential and k algebraic variables.

Special case 4. *Equations* (49i)-(49m) *can be solved symbolically.*

We can decouple these equations from the system, removing $j_1+j_2+\ldots+j_r+k$ algebraic variables and obtaining an ODE system given by equations (49a)-(49h). □

An example of a Hessenberg system of higher index falling under the Special case 4 is a high-order differential equation subject to algebraic constraints.

The analog of Special case 1 can be also considered for higher Hessenberg index DAEs. Then, following the guidelines given in the Special case 1, we can additionally eliminate l differential variables, where l is the number of linear constraints.

5. A Particular Case of DAEs of More General Form

In this section we consider a case when DAEs are not of the form (2), (4), or (6) and show how they can be reduced to one of these forms, in order to apply the projection method. We call this case *mixed Hessenberg index-1,3 form*.

We define the mixed Hessenberg index-1,3 form as a system of the form

$$\begin{aligned}
\dot{\mathbf{y}}&=\mathbf{f}(\mathbf{x},\mathbf{y},\mathbf{z}), && (50a)\\
\dot{\mathbf{x}}&=\mathbf{g}(\mathbf{x},\mathbf{y}), && (50b)\\
0&=\mathbf{h}(\mathbf{x})+\mathbf{A}(\mathbf{x})\mathbf{z}, && (50c)
\end{aligned}$$

where the matrix $\mathbf{A}$ is not invertible. We also define a solvability condition guaranteeing that system (50) has index 3.

In order to formulate the solvability condition, we introduce additional matrices. If the matrix $\mathbf{A}$ has rank $k_1<k$,

re-ordering its rows and columns if necessary, we can represent it as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}, \tag{51}$$

where

$$\text{rank } \mathbf{A} = \text{ rank } \mathbf{A}_{11} = k_1, \tag{52}$$

and $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{21}$, and $\mathbf{A}_{22}$ are, respectively, $k_1 \times k_1$, $k_1 \times (k - k_1)$, $(k - k_1) \times k_1$, and $(k - k_1) \times (k - k_1)$-dimensional matrices. The condition (52) yields that $\mathbf{A}_{11}$ is invertible.

We introduce matrices $\mathbf{L}$ and $\mathbf{R}$:

$$\mathbf{L} = \left(\frac{\mathbf{L}_1}{\mathbf{L}_2} \right) = \left(\begin{array}{cc} \mathbf{I}_{k_1} & \mathbf{0} \\ \hline -\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{I}_{k-k_1} \end{array} \right), \tag{53a}$$

$$\mathbf{R} = \left(\mathbf{R}_1 \mid \mathbf{R}_2 \right) = \left(\begin{array}{c|c} \mathbf{I}_{k_1} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \\ \mathbf{0} & \mathbf{I}_{k-k_1} \end{array} \right). \tag{53b}$$

Now we can formulate the solvability condition as follows:

$$\frac{\partial(\mathbf{L}_2\mathbf{h})}{\partial \mathbf{x}} \frac{\partial \mathbf{g}}{\partial \mathbf{y}} \frac{\partial \mathbf{f}}{\partial \mathbf{z}} \mathbf{R}_2 \text{ is invertible.} \tag{54}$$

In order to apply the projection method, we introduce a change of variables:

$$\mathbf{z} = \mathbf{R}\zeta. \tag{55}$$

Substituting transformation (55) into equation (50c) and multiplying it by $\mathbf{L}$ on the left, we obtain:

$$\mathbf{Lh} + \mathbf{LAR}\zeta = 0. \tag{56}$$

From definitions (53) it follows that

$$\mathbf{LAR} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Let us denote the first k_1 rows of ζ by ζ_1 and the last $k - k_1$ by ζ_2 and consider the first k_1 rows of equation (56):

$$\mathbf{L}_1\mathbf{h} + \mathbf{A}_{11}\zeta_1 = 0. \tag{57}$$

From condition (52) it follows that equation (57) can be solved for ζ_1, obtaining:

$$\zeta_1 = -\mathbf{A}_{11}^{-1}\mathbf{L}_1\mathbf{h}. \tag{58}$$

The last $k - k_1$ rows of equation (56) yield the following.

$$\mathbf{L}_2\mathbf{h} = 0, \tag{59}$$

which we consider as a new algebraic constraint. Substituting transformation (55) and representation (58) into the right-hand side of equation (50a), we obtain:

$$\begin{aligned} \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{R}\zeta) &= \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{R}_1\zeta_1 + \mathbf{R}_2\zeta_2) \\ &= \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{R}_2\zeta_2 - \mathbf{R}_1\mathbf{A}_{11}^{-1}\mathbf{L}_1\mathbf{h}). \end{aligned}$$

We obtain a Hessenberg index-3 DAE system

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{R}_2\zeta_2 - \mathbf{R}_1\mathbf{A}_{11}^{-1}\mathbf{L}_1\mathbf{h}), \tag{60a}$$

$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, \mathbf{y}), \tag{60b}$$

$$0 = \mathbf{L}_2(\mathbf{x})\mathbf{h}(\mathbf{x}). \tag{60c}$$

From the solvability condition (54) it follows that condition (7) on the product of the Jacobians is satisfied, and the system (60) is indeed of a Hessenberg index-3 DAE form. The projection method can now be applied to system (60).

We also consider a special case of mixed Hessenberg index-1,3 form, where the matrix $\mathbf{A}$ can be represented, by re-ordering columns and rows if necessary, in the form

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{61}$$

where $\mathbf{A}_{11}$ is invertible. Equations (50) now take the form

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \mathbf{z}_2), \tag{62a}$$

$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, \mathbf{y}), \tag{62b}$$

$$0 = \mathbf{h}_1(\mathbf{x}) + \mathbf{A}_{11}(\mathbf{x})\mathbf{z}_1 + \mathbf{A}_{12}(\mathbf{x})\mathbf{z}_2, \tag{62c}$$

$$0 = \mathbf{h}_2(\mathbf{x}), \tag{62d}$$

assuming the solvability condition is satisfied:

$$\frac{\partial \mathbf{h}_2}{\partial \mathbf{x}} \frac{\partial \mathbf{g}}{\partial \mathbf{y}} \frac{\partial \mathbf{f}}{\partial \mathbf{z_2}} \text{ is nonsingular for all } t.$$

In this case we do not need to define matrices $\mathbf{L}$ and $\mathbf{R}$. Our goal is to obtain index-1 DAEs, and the equations (62c) are already in index-1 form. We then apply the projection method to the system (62a), (62b), and (62d), treating the variables $\mathbf{z}_1$ as known, to obtain index-1 equations for $\mathbf{y}$ and $\mathbf{z}_2$ and obtain an index-1 system of DAEs consisting of the result of application of the projection method and (62c).

6. Optimization Under Constraints

We now show how a general optimization problem can be represented as a higher-index Hessenberg system, and how the projection method can then be applied.

Consider the problem of optimizing the functional

$$F(\mathbf{x}) = \int_{t_1}^{t_2} \mathcal{L}\left(t, \mathbf{x}, \dot{\mathbf{x}}, \ldots, \delta^k\mathbf{x}\right) dt, \tag{63}$$

where $\delta^k\mathbf{x} = \frac{\partial^k \mathbf{x}}{\partial t^k}$, subject to the constraints

$$\phi(\mathbf{x}) = \int_{t_1}^{t_2} \mathbf{h}(t, \mathbf{x}) dt = 0. \tag{64}$$

$\mathcal{L}$ is called a *Lagrangian*, $\mathbf{h} = (h_1, \ldots, h_m)^T$, $\phi = (\phi_1, \ldots, \phi_m)^T$, and $\phi_j = \int_{t_1}^{t_2} h_j(\mathbf{x}(t))dt$ for $j = 1, \ldots m$.

For example, a Lagrangian for a particle of charge q and mass m in an electromagnetic field with characteristics $\mathbf{E}$ and $\mathbf{B}$ is given by

$$\mathcal{L} = \frac{1}{2} m\dot{\mathbf{x}}^T\dot{\mathbf{x}} + q\dot{\mathbf{x}}^T\mathbf{A}(t, \mathbf{x}) - q\Phi(t, \mathbf{x}), \tag{65}$$

where $\mathbf{x}$ is the position of the particle. The scalar potential Φ and the vector potential $\mathbf{A}$ are such that

$$\mathbf{B} = \text{curl } \mathbf{A}, \quad \mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t},$$

where curl $\mathbf{A}$ is the curl of $\mathbf{A}$, and $\nabla\Phi$ is the gradient of Φ.

For the case of several particle charges, a Lagrangian is obtained as a sum of the Lagrangians of all particles. There is also a relativistic counterpart of Lagrangian (65) [4].

The variational principle for an electric circuit arises from imposing that the energy losses must be as small as possible [4]. A Lagrangian for interconnected electrical circuits consisting of linear elements can be written as follows [13]

$$\mathcal{L} = \frac{1}{2}\sum_{j,k} M_{jk}\dot{I}_j\dot{I}_k - \frac{1}{2}\sum_j \frac{1}{C_j} I_j^2 + \sum_j \dot{E}_j\dot{I}_j, \quad (66)$$

where I_j is the current through the j-th element, M_{jk} is the mutual inductance between the j-th and k-th inductors, C_j is the capacitance of the j-th capacitor, and E_j is the j-th electromotive force. Lagrangian (66) describes the energy exchange between the elements.

Resistors in an electric circuit act as generalized dissipative forces Q_j described by the function

$$W = \frac{1}{2}\sum_j R_j \dot{I}_j^2,$$

where

$$Q_j = -\frac{\partial W}{\partial \dot{I}_j} = -R_j\dot{I}_j$$

and R_j is the resistance of the j-th resistor.

Then the energy conservation law states that the loss of energy described by $\mathcal{L}$ is equal to the work done by the dissipative forces to create a change δI_j.

Lagrangian (66) can be also used to describe a mechanical system of masses connected by springs if parameters are defined as following: I_j is the displacement of the j-th mass, M is the reciprocal mass tensor, $1/C_j$ is the spring constant of the j-th spring, $\dot{E}_j$ is the j-th driving force, R_j is the viscous force of the j-th damper [13], and W is the Rayleigh dissipation function.

Optimal control problems can be formulated as problems of maximizing a functional using the Pontrjagin maximum principle [5].

There is no unique, general, automatic method for determining a Lagrangian for a physical system. Typically, the derivation of a Lagrangian is based on the invariance properties of the system [13]. For example, a free particle in a gravitational field is invariant under changes of time $t \to t + t_0$ without any change in coordinates, i.e. time has no privilege of origin, and, therefore, $\partial\mathcal{L}/\partial t = 0$. Space also has no privilege of origin for the particle, and, therefore, $\partial\mathcal{L}/\partial x_i = 0$. Finally, rotation invariance of the particle implies that $\mathcal{L}$ can only depend on the square of the velocity, i.e., it is of the form $\mathcal{L}(\dot{\mathbf{x}}^2)$ [4].

To tackle optimization problem (63), (64), we formulate the necessary condition for a functional $\mathcal{F}(\mathbf{x})$ subject to constraints $\psi(\mathbf{x}) = \mathbf{y}_0$ to have an extremum at $\mathbf{x}_0(t)$ [9].

THEOREM 2. *Let $U \subset \mathbb{R}^n$ be open and $\mathcal{F} : U \to \mathbb{R}$ a differential functional; furthermore, let $\psi : U \to \mathbb{R}^m$ be a continuously differential mapping. Let the functional $\mathcal{F}$ attain a local extremum at a regular point $\mathbf{x}_0 \in U$ of the mapping ψ subject to the condition $\psi(\mathbf{x}) = \mathbf{y}_0$. Then there exist real numbers $\lambda_1, \dots \lambda_m$ such that*

$$\frac{\partial \mathcal{F}}{\partial x_i}(\mathbf{x}_0) = \sum_{j=1}^m \lambda_j \frac{\partial \psi_j}{\partial x_i}(\mathbf{x}_0), \quad i = 1, \dots, n,$$

where all derivatives are understood in the Fréchet sense.

From Theorem 2 it follows that to find the extreme points of functional (63) under constraints (64), we need to solve the following system of DAEs:

$$\frac{\partial\mathcal{L}}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial \dot{x}_i}\right) + \dots + (-1)^k\frac{d^k}{dt^k}\left(\frac{\partial\mathcal{L}}{\partial\left(\delta^k x_i\right)}\right) = \sum_{j=1}^m \lambda_j\frac{\partial h_j}{\partial x_i}, \quad 1 \leqslant i \leqslant n, \quad (67a)$$

$$h_j(x_1, \dots, x_n, t) = 0, \qquad 1 \leqslant j \leqslant m. \quad (67b)$$

For Lagrangian (66) describing an electric circuit in the absence of dissipative forces, equations (67) take the form

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{I}_i}\right) - \frac{\partial\mathcal{L}}{\partial I_i} = \sum_{j=1}^m \lambda_j\frac{\partial h_j}{\partial x_i}, \quad 1 \leqslant i \leqslant n.$$

In the presence of dissipative forces the energy loss is equal to the work done by the dissipative forces

$$\frac{\partial F}{\partial I_j} = \int_{t_1}^{t_2} -\frac{\partial W}{\partial \dot{I}_j}\delta I_j$$

and it has to be as small as possible. Then equations (67) become:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{I}_i}\right) - \frac{\partial\mathcal{L}}{\partial I_i} + \frac{\partial W}{\partial\dot{I}_j} = \sum_{j=1}^m \lambda_j\frac{\partial h_j}{\partial x_i}, \quad 1 \leqslant i \leqslant n. \quad (68)$$

We can re-write equation (67) as:

$$\mathbf{M}(t, \mathbf{x}, \dot{\mathbf{x}}, \dots, \delta^{2k-1}\mathbf{x})\delta^{2k}\mathbf{x} = \mathbf{f}(t, \mathbf{x}, \dot{\mathbf{x}}, \dots, \delta^{2k-1}\mathbf{x}) + \mathbf{C}^T\lambda, \quad (69a)$$

$$\mathbf{h}(t, \mathbf{x}) = 0, \quad (69b)$$

where $\mathbf{C} = \partial\mathbf{h}/\partial\mathbf{x}$, $\lambda = (\lambda_1, \dots \lambda_m)^T$, and $\mathbf{f}$ is a function of the derivatives of $\mathbf{x}$ of order $< 2k$. System (69) is an instance of Special case 4 from Section 4. In order to see this we re-write system (69) as a system of first-order DAEs. Since we know that in Special case 4 we obtain a significant simplification by applying the projection method, we follow the steps in Section 4 and introduce a set of $(n-m)$-dimensional variables $\mathbf{u}_1, \dots, \mathbf{u}_{2k-1}$ such that

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{D}\mathbf{u}_1 + \mathbf{C}^T\mathbf{v}, \\ \dot{\mathbf{u}}_1 &= \mathbf{u}_2, \\ &\vdots \\ \dot{\mathbf{u}}_{2k-2} &= \mathbf{u}_{2k-1}. \end{aligned} \quad (70)$$

Then equation (69) can be rewritten as

$$\widetilde{\mathbf{M}}\mathbf{D}\dot{\mathbf{u}}_{2k-1} = \widetilde{\mathbf{f}} + \mathbf{C}^T\lambda - \widetilde{\mathbf{M}}\widetilde{\sigma} - \widetilde{\mathbf{M}}\delta^{2k-1}\left(\mathbf{C}^T\mathbf{v}\right), \quad (71)$$

where $\mathbf{D}$ is such that $\mathbf{CD} = 0$,

$$\begin{aligned} \mathbf{v} &= -\left(\mathbf{C}\mathbf{C}^T\right)^{-1}\frac{\partial\mathbf{h}}{\partial t}, \\ \widetilde{\mathbf{M}} &= \widetilde{\mathbf{M}}(t, \mathbf{x}, \mathbf{u}_1, \dots, \mathbf{u}_{2k-1}) \\ &= \mathbf{M}(t, \mathbf{x}, \mathbf{D}\mathbf{u}_1 + \mathbf{C}^T\mathbf{v}, \dots, \delta^{2k-2}(\mathbf{D}\mathbf{u}_1 + \mathbf{C}^T\mathbf{v})), \end{aligned}$$

$$
\begin{aligned}
\widetilde{\mathbf{f}} &= \widetilde{\mathbf{f}}(t,\mathbf{x},\mathbf{u}_1,\dots,\mathbf{u}_{2k-1})\\
&= \mathbf{f}(t,\mathbf{x},\mathbf{D}\mathbf{u}_1+\mathbf{C}^T\mathbf{v},\dots,\delta^{2k-2}(\mathbf{D}\mathbf{u}_1+\mathbf{C}^T\mathbf{v})),\\
\widetilde{\sigma} &= \widetilde{\sigma}(t,\mathbf{x},\mathbf{u}_1,\dots,\mathbf{u}_{2k-1})\\
&= \delta^{2k-1}(\mathbf{D}\mathbf{u}_1) - \mathbf{D}\delta^{2k-1}\mathbf{u}_1
\end{aligned}
$$

and we have replaced the derivatives of $\mathbf{u}_1$ with their expressions in terms of $\mathbf{u}_2,\dots,\mathbf{u}_{2k-1}$ according to (71).

Now we multiply equation (71) by $\mathbf{D}^T$ from the left and, combining with (70), get the following simplified system:

$$\dot{\mathbf{x}} = \mathbf{D}\mathbf{u}^{(1)} + \mathbf{C}^T\mathbf{v}, \tag{72a}$$

$$\dot{\mathbf{u}}^{(1)} = \mathbf{u}^{(2)}, \tag{72b}$$

$$\vdots$$

$$\dot{\mathbf{u}}^{(2k-2)} = \mathbf{u}^{(2k-1)}, \tag{72c}$$

$$\dot{\mathbf{u}}^{(2k-1)} = \left(\mathbf{D}^T\widetilde{\mathbf{M}}\mathbf{D}\right)^{-1}\mathbf{D}^T\left[\widetilde{\mathbf{f}} - \widetilde{\mathbf{M}}\widetilde{\sigma} - \widetilde{\mathbf{M}}\delta^{2k-1}\left(\mathbf{C}^T\mathbf{v}\right)\right], \tag{72d}$$

System (69) is equivalent to a system of DAEs with respect to $2kn$ differential and m algebraic variables. System (72), obtained by application of the projection method to (69), is an ODE system with respect to $n + (2k-1)(n-m)$ variables $(\mathbf{x},\mathbf{u}^{(1)},\mathbf{u}^{(2)},\dots,\mathbf{u}^{(2k-1)})$. Thus, we have eliminated $(2k-1)m$ differential and m algebraic variables.

Given the solution of equation (71) we can find the values of the Lagrange multipliers as

$$\lambda = \left(\mathbf{C}\widetilde{\mathbf{M}}^{-1}\mathbf{C}^T\right)^{-1}\mathbf{C}\left[\widetilde{\sigma} + \delta^{2k-1}\left(\mathbf{C}^T\mathbf{v}\right) - \widetilde{\mathbf{f}}\right].$$

For equations (72) we can also formulate an analog of the Special case 1 for Hessenberg index-3 DAEs. We repeat the derivations, since we will need the formulas in the example later.

Special case 5. *Some of the algebraic constraints are linear in* $\mathbf{x}$.

The function $\mathbf{h}$ is of the form

$$
\begin{aligned}
0 &= \mathbf{C}_1(t)\mathbf{x} + \alpha(t),\\
0 &= \mathbf{h}_2(t,\mathbf{x}),
\end{aligned}
$$

where $\mathbf{C}_1$ is an $l \times n$ matrix of maximal row-rank, such that $l \le n$, and $\mathbf{h}_2$ is a $(k-l)$-dimensional vector function which is nonlinear in $\mathbf{x}$. We can represent $\mathbf{x}$ as

$$\mathbf{x} = \mathbf{D}_1\chi + \mathbf{C}_1^T\psi, \tag{73}$$

where $\mathbf{D}_1$ is such that $\mathbf{C}_1\mathbf{D}_1 = 0$, $\chi = (\chi_1,\dots,\chi_{n-l})^T$, and $\psi = (\psi_1,\dots,\psi_l)^T$ is given by

$$\psi = -\left(\mathbf{C}_1\mathbf{C}_1^T\right)^{-1}\alpha.$$

Substituting representation (73) into equation (72a), we obtain an equation for $\dot{\chi}$:

$$\dot{\chi} = (\mathbf{D}_1\mathbf{D}_1)^{-1}\mathbf{D}_1^T\left[\mathbf{D}\mathbf{u}^{(1)} + \mathbf{C}^T\mathbf{v} - \dot{\mathbf{D}}_1\chi - \dot{\mathbf{C}}_1\psi\right]. \tag{74}$$

By replacing $\mathbf{x}$ with its representation (73) and equation (72a) with equation (74), we can eliminate l variables from ODEs (72). □

We now demonstrate the application of the projection method to the optimization of a functional with Lagrangian (66).

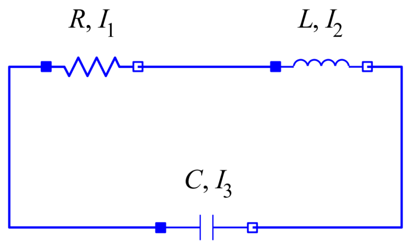

Figure 1. Serial *RLC* circuit.

7. Example: Electric Circuit

We consider an electric circuit consisting of a resistor of resistance R, an inductor of inductance L, and a capacitor of capacitance C connected in series as depicted in Figure 1.

From (66) a Lagrangian for the circuit is given by

$$\mathcal{L} = \frac{1}{2}L\dot{I}_1^2 - \frac{1}{2}\frac{1}{C}I_3^2 + \lambda_1(I_1 - I_2) + \lambda_2(I_2 - I_3), \tag{75a}$$

$$W = \frac{1}{2}R\dot{I}_1^2, \tag{75b}$$

where the constraints, $I_1 = I_2$ and $I_2 = I_3$, represent Kirchhoff's Current Law.

From equations (67) and (68), the necessary condition for a functional with $\mathcal{L}$ and W given by equations (75) takes form of the following DAE system:

$$RI_1 = \lambda_1, \tag{76a}$$

$$L\ddot{I}_2 = -\lambda_1 + \lambda_2, \tag{76b}$$

$$\frac{1}{C}I_3 = -\lambda_2, \tag{76c}$$

$$0 = I_1 - I_2, \tag{76d}$$

$$0 = I_2 - I_3, \tag{76e}$$

where equations (76d) and (76e) are as described in Special case 5. We define:

$$\mathbf{C} = \frac{\partial\mathbf{h}}{\partial\mathbf{x}} = \begin{pmatrix} 1 & -1 & 0\\ 0 & 1 & -1 \end{pmatrix}.$$

It is easy to check that $\mathbf{D} = (1,1,1)^T$. Note that the right-hand side of equations (76a), (76b), and (76e) is indeed given by $\mathbf{C}^T\lambda$. From equation (73)

$$I_1 = u^{(1)},\ I_2 = u^{(1)},\ I_3 = u^{(1)}, \tag{77}$$

where $u^{(1)}$ satisfies the system

$$\dot{u}^{(1)} = u^{(2)}, \tag{78a}$$

$$\dot{u}^{(2)} = -\frac{R}{L}u^{(2)} - \frac{1}{LC}u^{(1)} \tag{78b}$$

that follows from equation (72).

Equations (78) are the desired ODEs. After solving them, we find the current using (77).

8. Implementation and Benchmarks

We implemented our algorithm in the computer algebra system Maple (version 16). Table 2 gives some benchmarks for 5 DAEs of index 3 that were created using a conserved

Model	Version	DE	AE	SE	Time
DoublePendulum1	HF	22	7	112	
	PM	14	0	129	0.91s
FourBar	HF	30	11	166	
	PM	16	0	194	13.03s
Pendulum1	HF	12	3	58	
	PM	8	0	67	0.29s
SliderCrank	HF	29	10	215	
	PM	18	0	231	2.15s
TriplePendulum1	HF	32	11	165	
	PM	20	0	199	1.86s

Table 2. Implementation benchmarks

quantities based modeling tool, HLMT [14, 3], and then converted to (mixed) Hessenberg index-3 form. (Such models tend to be more verbose and contain more redundancy than DAEs that were created using other modeling tools.)

There are two rows per model. The first row lists the number of equations in the Hessenberg form that is input to our algorithm. The second row gives the number of equations in the resulting model after applying the projection method, as well as the running time (3 GHz Intel Core 2 Duo) of our implementation.

Three types of equations are counted in each row: the differential equations (DE) and the algebraic equations (AE) together form the *core dynamical system*. The solved equations (SE) are explicit algebraic equations expressing variables that do not appear in the core system exclusively in terms of core system variables. In all 5 examples, the projection method was able to remove all (possibly non-linear and implicit) algebraic constraints, thanks to Special cases 2 and 3 above.

9. Conclusions

We have derived how the projection method introduced in [16] for mechanical systems can be extended to other types of DAEs. The projection method not necessarily simplifies DAEs better than index reduction alone. However, in some cases the application of the projection method after index reduction can decrease the number of variables and even simplify the reduced index-1 DAE to an ODE. We have described these cases and demonstrated that optimization under constraints is one of them. We have shown the significance of optimization problems in various applications and derived an effective simplification procedure for the optimization of a functional under constraints. As an example, we have formulated a problem of current flowing through an electric circuit as an optimization problem and used it to illustrate the applicability of the projection method. Finally, we have given benchmarks for an implementation of the projection method in Maple.

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