The Wave Excitation Forces on a Floating Vertical Cylinder in Water of Infinite Depth

William Finnegan[1], Martin Meere[2], Jamie Goggins[3,*]

[1] *College of Engineering and Informatics, National University of Ireland, Galway, Ireland*
[2] *School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland*
[3] *Ryan Institute for Environmental, Marine and Energy Research, National University of Ireland, Galway, Ireland*
** Tel: +35391492609, E-mail: jamie.goggins@nuigalway.ie*

Abstract: When carrying out any numerical modeling it is vital to have an analytical approximation to insure that realistic results are obtained. The numerical modeling of wave energy converters is an efficient and inexpensive method of undertaking initial optimisation and experimentation. Therefore, the main objective of this paper is to determine an analytical solution for the heave, surge and pitch wave excitation forces on a floating cylinder in water of infinite depth. The boundary value problem technique, using the method of separation of variables, is employed to derive the velocity potentials throughout the fluid domain. A Fourier transform is used to represent infinite depth. Additionally, Havelock's expansion theorem is used to invert the complicated combined Fourier sine/cosine transform. An asymptotic approximation is taken for low frequency incident waves in order to create an analytical solution to the problem. Graphical representations of the wave excitation forces with respect to incident wave frequencies for various draft to radius ratios are presented, which can easily be used in the design of wave energy converters.

Keywords: *Infinite depth, Wave energy, Wave structure interaction, Wave water problem*

Nomenclature

a	*radius of cylinder*	m
A	*amplitude of incident wave*	m
b	*draft of cylinder*	m
F	*force*	N
F_c	*Fourier cosine transform*	
$F_{1,ext}$	*surge excitation force*	N
$F_{3,ext}$	*heave excitation force*	N
G	*gravity*	$m \cdot s^{-2}$
k_0	*wavenumber*	m^{-1}
m	*integer*	
n_j	*j-component of the normal*	
$p_m(\xi)$	*coefficient*	
q_{m0}	*coefficient*	
$q_m(\xi)$	*coefficient*	
r	*radius*	m
S_B	*wetted surface*	
t	*time*	
v	*flow velocity*	$m \cdot s^{-1}$
x	*horizontal coordinate*	m
z	*vertical coordinate*	m
ε_m	*Neumann symbol*	
θ	*polar coordinate*	*rads*
ξ	*separation constant*	
ρ	*density*	$kg \cdot m^3$
φ	*frequency domain velocity potential*	$m \cdot s^{-1}$
φ_I	*incident wave velocity potential*	$m \cdot s^{-1}$
φ_s^i	*interior scattering velocity potential*	$m \cdot s^{-1}$
φ_d^e	*exterior diffraction velocity potential*	$m \cdot s^{-1}$
φ_s^e	*exterior scattering velocity potential*	$m \cdot s^{-1}$
Φ	*time domain velocity potential*	$m \cdot s^{-1}$
ω	*wave angular frequency*	s^{-1}

1. Introduction

One of the main stages in the design of wave energy converters (WECs) is the numerical modelling of a given converter. In this paper, an analytical solution for the wave excitation forces on a floating cylinder in water of infinite depth is provided. The solution will act as a method of validating the results from numerical models of WECs, as it provides an estimation of forces on a cylinder representation of an arbitrary shaped axisymmetric WEC.

The solution of the scattering and radiation problem for floating bodies, in finite or infinite depth water, has being explored for decades for a various shapes of bodies. In 1948, Fritz Ursell[1] explored the forces on an infinitely long horizontal floating cylinder in infinitely deep water and, in 1955, Sir Thomas Havelock[2] solved the radiation problem for a floating half-immersed sphere in infinitely deep water. In 1971, Garrett[3] solved the scattering problem by determining the vertical force, horizontal force and torque for a circular dock in water of finite depth. In 1975, Black[4] looked at the wave forces on bodies which are vertically axisymmetric using an integral equation formulation in water of finite depth. In 1981, Yeung[5] presented a set of theoretical added mass and damping coefficient for a floating cylinder in finite depth, which he also truncated for the infinite depth problem. In 2003, Bhatta and Rahman[6] used a similar technique as Havelock to solve, although using a semi-analytic solution, the scattering and radiation problem for a floating vertical cylinder in water of finite depth. Previously, an analytical solution for wave excitation forces on a floating cylinder in water of infinite depth has not been derived. Therefore, the solution derived in this paper is for a semi-submerged vertical cylinder in infinite depth water. A boundary value problem is used to derive an analytic solution, from the scattering problem, for the heave, surge and pitch excitation forces.

2. Methodology

The problem considers a vertical cylinder, of radius, *a*, and with a draft, *b*, which can move in surge, heave or pitch motion, and an incident wave of amplitude, *A*, and angular frequency, ω, as depicted in Fig. 1. The wave progresses in the positive x-direction with the origin at the still water level (SWL) and the positive z-direction is vertically downwards. In the formulation of the solution, a number of assumptions are used:

- The water is both incompressible, as frequencies are low, and effectively viscid.
- As the air has such a small density, pressure change is negligible and, thus, is at constant pressure.
- The surface tension at air-water interface is negligible.
- The water is at constant density and temperature.
- The Reynolds' number for the flow is sufficiently small for the flow to remain laminar.
- The waves are progressive and only travel in one direction and the wave motion is irrotational.

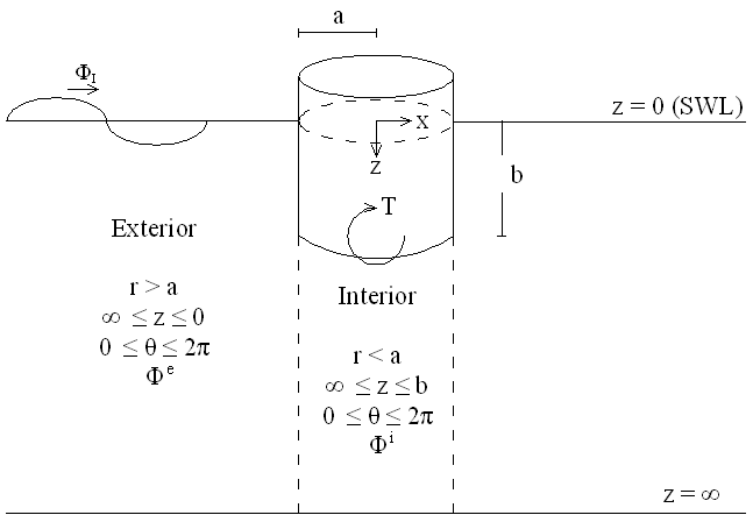

Fig. 1 Graphical set-up of the Boundary Value Problem for a Vertical Cylinder

Yeung[5] and Bhatta and Rahman[6] employed the technique of dividing the domain into two regions, which is used in this paper. The two regions are the interior region, which is the area underneath the cylinder, and the exterior region, which is the remaining area of the fluid (Fig. 1). The problem is solved in the frequency domain. Therefore, the velocity potential, φ, to be solved is transformed to the frequency domain, as follows:

$$\Phi(r,\theta,z,t) = \mathrm{Re}\{\varphi(r,z,\theta)e^{-i\omega t}\} \tag{1}$$

where Φ is the time domain velocity potential, r is radius, θ is the angle, i is the standard imaginary unit, ω is the wave angular frequency of the wave, and φ is the frequency domain velocity potential. The force is then calculated by integrating the velocity potential over the wetted surface area of the cylinder, S_B, using the following equation:

$$\hat{F} = i\rho\omega\int_{SB} \phi n_j \, dS \tag{2}$$

where ρ is water density, n_j is the j-component of the normal, S is surface and F is the force, where $F = \mathrm{Re}\{\hat{F}e^{-i\omega t}\}$. The equations and boundary conditions that need to be satisfied throughout the problem are: the Laplace's equation, the deep water condition, the free surface equation and the radiation condition, respectively[7]:

$$\square^2\varphi = \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \tag{3}$$

$$|\square\varphi| \to 0 \text{ as } z \to \infty \tag{4}$$

$$\omega^2\varphi - g\frac{\partial\varphi}{\partial z} = 0 \text{ on } z = 0,\ r \geq a \tag{5}$$

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial\varphi}{\partial r} - ik_0\varphi\right) = 0 \tag{6}$$

where k_0 is the wavenumber ($k_0 = \omega^2/g$). Since the motion is irrotational and incompressible, the Laplace's equation was arrived at by substituting $v = \square\varphi$ into $\square v = 0$, where v is the flow velocity. The solution being developed is for infinitely deep water. Thus, the deep water condition defines the flow velocity near the sea bed. The free surface equation defines the velocity potential at the free surface away from the floating body. The radiation condition defines the velocity potential of the wave at the distance from the body when the effect of the body on the wave has dissipated. The scattering problem deals with the excitation force on a fixed body and, therefore, the following structural boundary conditions must be imposed:

$$\frac{\partial\varphi_S^i}{\partial\bar{z}} = 0 \text{ on } \bar{z} = 0, \quad \text{where } \bar{z} = z - b \tag{7}$$

$$\frac{\partial\varphi_S^e}{\partial r} = 0 \text{ at } r = a \tag{8}$$

where φ_S^i and φ_S^e are the interior and exterior scattering velocity potentials, respectively. Since we are dealing with infinite depth, a Fourier sine/cosine transform is employed when dealing with the vertical or z-component. For the interior region, in order to satisfy the structural equation (Eq. (7)) a Fourier cosine transform is required. Therefore, introducing a constant, ξ, yields:

$$F_C\left(\varphi_S^i(r,\theta,\bar{z})\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \varphi_S^i(r,\theta,\bar{z})\cos\xi\bar{z}\,d\bar{z} \tag{9}$$

$$\square\ \varphi_S^i(r,\theta,\bar{z}) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c\left(\varphi_S^i(r,\theta,\bar{z})\right)\cos\xi\bar{z}\,d\xi \tag{10}$$

where F_c is the Fourier cosine transform. The method of separation of variables is used to solve the Laplace's equation (Eq. (3)) in order to formulate an expression for the interior scattering velocity potential φ_S^i, as follows:

$$\varphi_S^i(r,\theta,\bar{z}) = \sum_{m=0}^{\infty}\sqrt{\frac{2}{\pi}}\int_0^\infty p_m(\xi)\frac{I_m(\xi r)}{I_m(\xi a)}\cos\xi\bar{z}\,d\xi\cos m\theta \tag{11}$$

where I_m is the modified first Bessel function of order m and $p_m(\xi)$ is an unknown coefficient.

Kim[8] gives the incident wave velocity potential, φ_I, in the frequency domain for deep water in oblique sea as:

$$\varphi_I(r,\theta,z) = -\frac{gA}{\omega}e^{-k_0 z}\sum_{m=0}^{\infty}\varepsilon_m i^{m+1}J_m(k_0 r)\cos m\theta \tag{12}$$

where J_m is the first Bessel function of order m, ε_m is the Neumann symbol, defined by $\varepsilon_0 = 1$ and $\varepsilon_m = 2$ for $m \geq 1$. Similarly, for the exterior region, when dealing with infinite depth in the method of separation of variables, a Fourier sine/cosine transform is used. In order to satisfy the free surface equation (Eq. (5)), a combination of the Fourier sine and Fourier cosine transform is required. Again, introducing a constant ξ, the following is obtained:

$$F\left(\varphi_d^e(r,\theta,z)\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \varphi_d^e(r,\theta,z)\left[\xi\cos\xi z - k_0\sin\xi z\right]\cos\xi z\,dz \tag{13}$$

where φ_d^e is the exterior diffraction velocity potential. The Havelock's expansion theorem [9] is used to obtain the inverse Fourier transform. Similarly, the method of separation of variables is used to solve the Laplace's equation (Eq. (3)) in order to formulate an expression for the exterior diffraction velocity potential, which is given as:

$$\begin{aligned}\varphi_d^e(r,\theta,z) = \sum_{m=0}^{\infty}[q_{m,0}\frac{H_m^{(1)}(k_0 r)}{H_m^{(1)}(k_0 a)}e^{-k_0 z} \\ + \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{q_m(\xi)}{\xi^2 + k_0^2}\frac{K_m(\xi r)}{K_m(\xi a)}\left[\xi\cos\xi z - k_0\sin\xi z\right]d\xi]\cos m\theta\end{aligned} \tag{14}$$

Therefore, since the scattering velocity potential is the sum of the incident and diffraction velocity potentials (i.e. $\varphi_S = \varphi_I + \varphi_d$) and incorporating $-gA\omega^{-1}\varepsilon_m i^{m+1}$ into the $\varphi_d^e(r,\theta,z)$ term in Eq. (14), the scattering velocity potential for the exterior problem is given as:

$$\varphi_S^e(r,\theta,z) = \sum_{m=0}^{\infty} -\frac{gA}{\omega}\varepsilon_m i^{m+1}[\{J_m(k_0 r) + q_{m,0}\frac{H_m^{(1)}(k_0 r)}{H_m^{(1)}(k_0 a)}\}e^{-k_0 z} + \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{q_m(\xi)}{\xi^2 + k_0^2}\frac{K_m(\xi r)}{K_m(\xi a)}[\xi\cos\xi z - k_0\sin\xi z]d\xi]\cos m\theta \tag{15}$$

where $H_m^{(1)}$ is the first Hankel function of order m and K_m is the modified second Bessel function of order m. The unknown coefficients of $p_m(\xi)$ in Eq. (11), and $q_{m,0}$ and $q_m(\xi)$ in Eq. (15), are found by matching the velocity potentials across the boundary at $r = a$. The conditions which are to be satisfied at the boundary are:

$$\varphi_S^e(r,\theta,z) = \varphi_S^i(r,\theta,\bar{z}), \text{ if } b \le z \le \infty \tag{16}$$

$$\frac{\partial\varphi_S^e(r,\theta,z)}{\partial r} = \frac{\partial\varphi_S^i(r,\theta,\bar{z})}{\partial r}, \text{ if } b \le z \le \infty \tag{17}$$

$$\frac{\partial\varphi_S^e(r,\theta,z)}{\partial r} = 0, \text{ if } 0 \le z \le b \tag{18}$$

3. Results

In order to create an analytical solution, asymptotic approximations for the excitation forces are derived for low frequency waves or, in other terms, when the wavenumber, k_0, tends towards zero. Therefore, in addition to Eq. (16)-(18), the approximation that k_0 tends towards zero is imposed when matching the interior scattering velocity potential, given in Eq. (11), and the exterior scattering velocity potential, given in Eq. (15), across the boundary $r = a$ in order to solve for the unknown coefficients $p_m(\xi)$, $q_{m,0}$ and $q_m(\xi)$,. Using this additional approximation, it was found that $q_m(\xi)$ tends to zero and the coefficient, $q_{m,0}$, is approximated as:

$$q_{m,0} = -J_m'(k_0 a)\frac{H_m^{(1)}(k_0 a)}{H_m^{(1)\prime}(k_0 a)} \tag{19}$$

and the coefficient, $p_m(\xi)$, is given as:

$$p_m(\xi) = -\frac{gA}{\omega}\varepsilon_m i^{m+1}\sqrt{\frac{2}{\pi}}\{J_m(k_0 a) + J_m'(k_0 a)\frac{H_m^{(1)}(k_0 a)}{H_m^{(1)\prime}(k_0 a)}\}\frac{e^{-k_0 b}k_0}{\xi^2 + k_0^2} \tag{20}$$

where prime is the derivative. Therefore, an analytical approximation is created and shown graphically for various draft, b, to radius, a, ratios in Fig. 2-4.

When calculating the surge, or horizontal, excitation force the only non-zero solution is when $m = 1$, as this is the only non-zero solution to the integral $\int cosm\theta\ cos\theta\ d\theta$, which arises in the force calculation. Furthermore, when integrating the velocity potential over the surface area,

the integration is performed only over the curved surface of the cylinder and, hence, the exterior velocity potential at $r = a$ is used. Therefore, the surge excitation force, $\hat{F}_{1,ext}$, is given as:

$$\begin{aligned}\hat{F}_{1,ext} &= i\rho\omega\int_{SB} \phi_S^e(r,\theta,z)n_1\, dS = i\rho\omega\int_0^{2\pi}\int_0^b \varphi_S^e(a,\theta,z)n_1\, a\, dz\, d\theta \\ &= -\frac{\rho g A a}{k_0}\sum_{m=0}^{\infty}\varepsilon_m i^m \{J_m(k_0a) - J_m'(k_0a)\frac{H_m^{(1)}(k_0a)}{H_m^{(1)\prime}(k_0a)}\}(1-e^{-k_0b})\int_0^{2\pi}\cos m\theta\cos\theta\, d\theta \qquad (21) \\ &= -\frac{2\pi i\rho g A a}{k_0}\{J_1(k_0a) - J_1'(k_0a)\frac{H_1^{(1)}(k_0a)}{H_1^{(1)\prime}(k_0a)}\}(1-e^{-k_0b})\end{aligned}$$

where $n_1 = -\cos\theta$. Graphical representations of surge excitation forces with respect to incident wave frequencies for various draft to radius ratios of devices are shown in Fig. 2.

When calculating the heave, or vertical, excitation force from the velocity potential, the only non-zero solution is when m is equal to zero due to the integral $\int \cos m\theta\, d\theta$. Furthermore, when integrating the velocity potential over the surface area, the integration is performed only over the base of the cylinder and, hence, the interior velocity potential, at $\bar{z} = 0$, is used. Therefore, the heave excitation force, $\hat{F}_{3,ext}$, is given as:

$$\begin{aligned}\hat{F}_{3,ext} &= i\rho\omega\int_{SB} \phi_S^i(r,\theta,\bar{z})n_3\, dS = i\rho\omega\int_0^{2\pi}\int_0^a \varphi_S^i(r,\theta,0)n_3\, r\, dr\, d\theta \\ &= -i\rho\omega\sqrt{\frac{2}{\pi}}\int_0^{2\pi}\int_0^a\int_0^{\infty} p_m(\xi)\frac{I_m(\xi r)}{I_m(\xi a)}d\xi\, r\, dr\cos m\theta\, d\theta \qquad (22) \\ &= -2\pi i\rho\omega a\sqrt{\frac{2}{\pi}}\int_0^{\infty} p_0(\xi)\frac{I_1(\xi r)}{\xi I_0(\xi a)}d\xi\end{aligned}$$

where $n_3 = -1$. Graphical representations of heave excitation forces with respect to incident wave frequencies for various draft to radius ratios of devices are shown in Fig. 3.

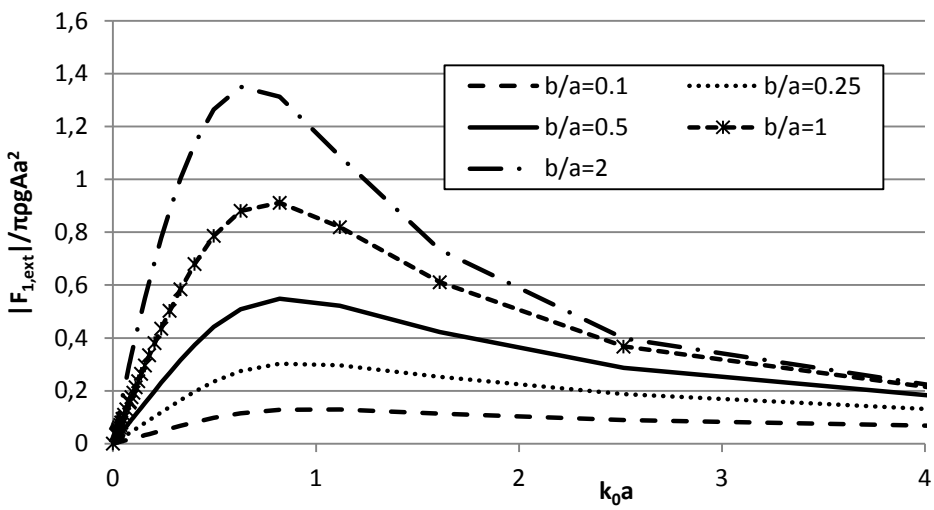

Fig. 2 The normalised surge (or horizontal) excitation force, in the frequency domain, as a function of k_0a for various radius to draft ratios.

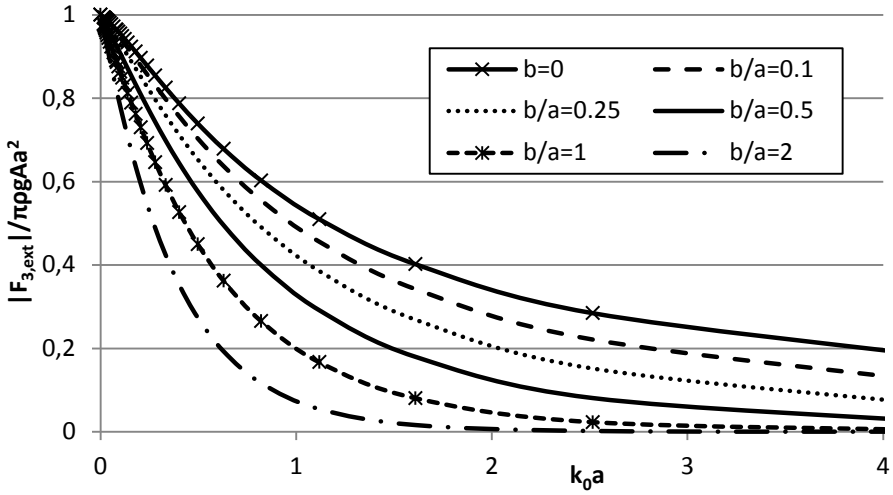

Fig. 3 The normalised heave (or vertical) excitation force, in the frequency domain, as a function of k_0a for various draft to radius ratios.

The pitch, or torque, excitation force arises from the surge and heave forces on the wetted surface of the cylinder. The pitch is taken about the axis which is transverse to the incident wave at the centre of the base, as shown by *T* in Fig. 1. When calculating the pitch the only non-zero solution, similar to surge, is when $m = 1$. Therefore, the pitch excitation force, $\hat{F}_{5,ext}$, is given as:

$$\begin{aligned}\hat{F}_{5,ext} &= i\rho\omega\int_{SB} \phi_S(r,\theta,z)n_5\, dS \\ &= -i\rho\omega\int_0^{2\pi}\int_0^b \varphi_S^e(a,\theta,z)(z-b)\cos\theta\, a\, dz\, d\theta + i\rho\omega\int_0^{2\pi}\int_0^a \varphi_S^i(r,\theta,0)r^2\cos\theta\, dr\, d\theta \\ &= -2\pi i\rho gAa\{J_m(k_0a) - J_m'(k_0a)\frac{H_m^{(1)}(k_0a)}{H_m^{(1)\prime}(k_0a)}\}\int_0^b (z-b)e^{-k_0z}\, dz \\ &+ \pi i\rho\omega a^2\sqrt{\frac{2}{\pi}}\int_0^\infty p_1(\xi)\frac{I_2(\xi r)}{\xi I_1(\xi a)}d\xi\end{aligned} \tag{28}$$

Graphical representations of pitch excitation forces with respect to incident wave frequencies for various draft to radius ratios of devices are shown in Fig. 4.

4. Discussion and Conclusions

An analytical solution to determine the heave, surge and pitch wave excitation forces on a floating cylinder in water of infinite depth has been presented in this paper. For ease of use in the design of wave energy converters, a graphical representation of the wave excitation forces with respect to the incident wave frequencies for various draft to radius ratios of devices are given. In particular, the heave, surge and pitch excitation forces, which are the only three forces on an axisymmetric device, were derived. The analytical solutions were obtained using an asymptotic approximation for low frequency incident waves.

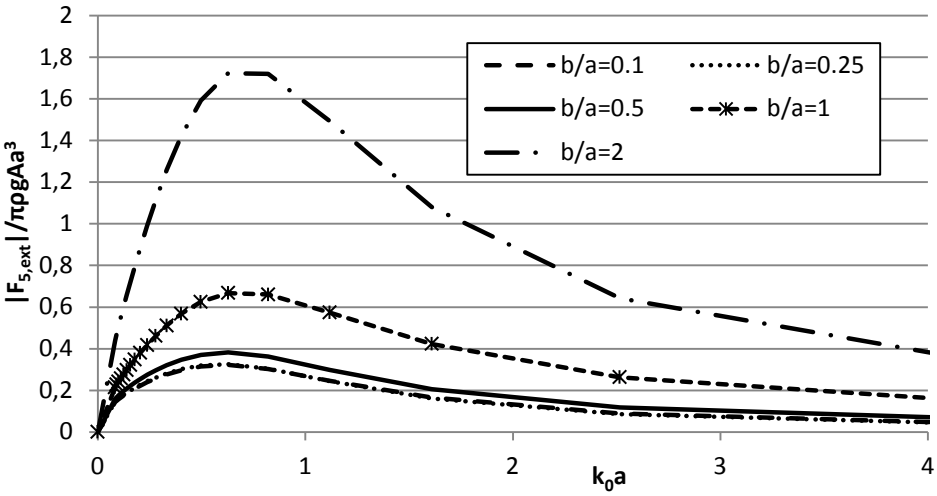

Fig. 4 The normalised pitch (or torque) excitation force, in the frequency domain, as a function of k_0a for various draft to radius ratios.

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