Actions, Ramification and Linear Modalities*

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0 Notation

We will, unfortunately, have to consider rather a lot of systems, and some uniform notation will be a help. LL will be classical linear logic; $\mathsf{LL}_\Diamond$ will be LL extended with modal operators of a certain sort. Typically, we will be given a rewrite system $\leadsto$, and we will be using the modal operators of $\mathsf{LL}_\Diamond$ to represent it; given a rewrite system, we will define two inequivalent modal systems, $\mathsf{LL}_\Diamond^{\leadsto}0$ and $\mathsf{LL}_\Diamond^{\leadsto}1$, which thus represent it. We will also want to detect when a sequence of rewrites terminates; this will involve an extension of $\mathsf{LL}^{\rightsquigarrow}_{\Diamond}1$ by two more operators, giving a system $\mathsf{LL}^{\rightsquigarrow}_{\Diamond,\boldsymbol{\sigma},\boldsymbol{\tau}}$.

Finally, given a system S, we will typically distinguish different presentations of it by primes, thus: S, S', S'', $\ldots$. (Terms such as S can thus stand both for systems and presentations of systems; this ambiguity will not cause any problems in practice.)

Sequent calculus rules may take different forms in different systems; when (and *only* when) this may cause confusion we will distinguish them by adding the name of the system in parentheses. For example, we will distinguish between

$$\frac{\Gamma \vdash A,\, \Delta}{\Gamma \vdash \Diamond A,\, \Delta}\,\Diamond\mathrm{R}(\mathsf{LL}_{\Diamond})$$

and

$$\frac{\Gamma \vdash A_1,\, \Delta}{\Gamma \vdash \Diamond A,\, \Delta}\,\Diamond\mathrm{R}(\mathsf{LL}^{\rightsquigarrow}_{\Diamond}1)$$

1 Outline

We should first recall the use of linear logic in [29, 31, 32]; the first two papers used linear logic to represent instances of the frame problem, and showed how proof search in linear logic could be used to give a computationally efficient algorithm for solving the frame problem. Finally, [32] showed how this approach could be extended to deal with complex actions, describing a system which could bisimulate Golog.

We would like, then, to start with this basic approach and integrate it with an account of ramification. Now since we can already simulate Golog, we could, in theory, write down complex actions which would yield the desired effects. However, we could justifiably have reservations about such a solution of the ramification problem; the notion of ramification seems to involve a certain division of responsibility between agent and world, and we want our solution to respect that division of responsibility. So we must first describe the problem phenomenologically, and only then should we try to formalise it.

1.1 Phenomenology

First, then, we need some sort of technically noncommittal, phenomenological account of ramification; we shall mostly follow Sandewall. His basic intuition is as follows:

> Suppose the world is in a stable state r, and an action E is invoked. The immediate effect of this is to set the world in a new state, which is not necessarily stable. If it is not, then one allows the world to go through the necessary sequence of state transitions, until it reaches a stable state. *That whole sequence* of state transitions is together viewed as the action, and the resulting admitted state is viewed as the result state of the action. [22, p. 15]

This picture of change has the following consequences for the treatment of action; they can be regarded as a policy of modularisation, of being able to divide the theoretical responsibility of representing the effects of actions into conceptually distinct components.

Firstly, since the result of change is, as it were, a joint product of the agent and the world, any formalisation should locate the properties of the world's response – which may be common to many actions – in the world-related part of the formalism:

> If some regularities in the world at hand are reflected in the consequent side of several of the action laws, then it makes sense to factor out the common parts and to represent them once and for all as "domain constraints". Instead of just applying the action law to determine the effects of an action, one applies the action law plus the domain constraints. [22, p. 11]

Secondly, our description of the world's response should likewise admit a corresponding decomposition:

> Consider applications where every scenario describes a number of separate but interconnected objects, different scenarios involve different configurations, and each action has its immediate effects on one or a few of the objects, but indirect effects on objects which are connected to the first ones, in some sense of the word 'connected'. Then, it would be completely unreasonable to let the action laws contain different cases which enumerate the possible configurations. Instead, action laws should only specify the "immediate" or "primary" effects of the action, and logical inference should be used for tracing how some changes "cause" other changes across the structure of the configuration at hand. [22, p. 11]

Sandewall's program is as follows: he has a formal semantics based on the basic intuition (it is, basically, described in terms of labelled transition systems) [22, Section 5]. He then evaluates various proposed solutions for the ramification problem against this underlying semantics. We will follow a similar path: first we will describe our approach to ramification, then we will compare it with Sandewall's basic intuition.

First some notation. States will be written as $\sigma_1, \sigma_2, \ldots$ and so on; the transitions that Sandewall talks of will be written as $\sigma_1 \rightsquigarrow \sigma_2$. We will suppose that we are given an account of which ramification transitions are possible (that is, we are given the relation $\sigma_1 \rightsquigarrow \sigma_2$) what we are lacking is a representation of all this in a suitable formal system.

1.2 Proposed Treatment

We recall [29, 32, 31] that we have a treatment of actions in linear logic according to which executions of an action α in a state σ_1, resulting in a state σ_2, correspond to linear logic proofs of the sequent

$$\sigma'_1,\, \alpha' \;\vdash\; \sigma'_2$$

where σ'_1, σ'_2, and α' are suitable linear logic formulae.

We would like to be able to accommodate ramification by extending this formalism. In line with our discussion of the phenomena, this extension should lie in the world-related part of our formalism, that is, in the formulae σ_1' and σ_2', which represent states.

Furthermore, we should notice that Sandewall's description is in terms of sequences of state transitions. The set of such sequences has two properties: it is closed under composition, and it contains the identity. More abstractly formulated, we will have what can be described semantically as a monad, or proof-theoretically as an S4 modal operator. We should recall that S4 modalities are given by the rules in Table 1; classically or intuitionistically, these rules give the usual modal logic [28, Section 9.1], but they can equally well be added to linear logic and they satisfy the usual proof-theoretic properties (cut elimination and so on) [14]. They are, in fact, somewhat like the linear logic exponentials, but without the contraction and weakening rules. And it seems, then, that we could have a linear S4 modality – call it $\Diamond$ – such that transitions from a state A to a state B would correspond to entailments of the form $A' \vdash \Diamond B'$. The fact that the allowable transitions contained the identity would correspond to the validity of $A \vdash \Diamond A$, for any A; and the closure of the allowable transitions under composition would correspond to the provability of $\Diamond\Diamond A \vdash \Diamond A$. We are thus starting from our transition relation, $\rightsquigarrow$, and we want to represent it as provability in linear logic enriched with suitable modalities.

So, if we had such a modality available, we could plug it in to the above treatment of action, and express the fact that an action was performed in a state s_1 leading – after ramification – to a state s_2 as the validity of a sequent

$$\sigma_1',\, \alpha' \;\vdash\; \Diamond\sigma_2'.$$

This is, perhaps, not so unexpected; there is a certain history of using modal operators on problems like these. McCarthy and Hayes [15, p. 472] *do* propose a reading of modal operators in terms of rewrites, but their modalities are normal Kripkean ones in a classical theory, and they are unable to make a very precise application of the modal logic. Mads Dam [3] defines a modal system, in the style of dynamic logic, extending positive linear logic. His modalities are emphatically not the same as ours; for example, his validate (in our notation) $\Diamond(A \oplus B) \vdash \Diamond A \oplus \Diamond B$ which, as we shall see, ours do not. Dam also has a somewhat different agenda to ours: he wants to find a logic whose semantics is given by traces of a certain process algebra, up to testing equivalence. This seems to be a delicate matter, and he consequently has to limit himself to rather weak systems.

1.2.1 Strong Modalities

Let us, then, consider modal operators in linear logic. Semantically, the $\Diamond$ looks very like a monad: that is, it is functorial (from $A \vdash B$ follows $\Diamond A \vdash \Diamond B$), it has a unit (that is, an entailment $A \vdash \Diamond A$ for any A), and it is idempotent (we have, for any A, an entailment $\Diamond\Diamond A \vdash \Diamond A$. We also need an implication

$$\Diamond(A \vee B) \vdash (\Diamond A) \vee (\Diamond B), \qquad (1)$$

for any A and B. Given this, we get the usual classical sequent calculus rules.

Now we need the entailment (1) in order to be able to derive $\Diamond A \vdash \Diamond B,\, \Diamond C$ from an entailment $A \vdash B,\, C$ (functoriality would only give us $\Diamond A \vdash \Diamond(B \vee C)$). In linear logic, we have both additive and multiplicative connectives: the contexts on either side of a sequent are formed using the multiplicative connectives (that is, the sequent $\Gamma \vdash \Delta$ is morally the same as $\bigotimes_{\gamma\in\Gamma} \gamma \vdash \bigparr_{\delta\in\Delta} \delta$), so we need to know something about how the operators interact with these connectives. We could consider the natural replacement for (1), that is

$$\Diamond(A \parr B) \vdash (\Diamond A) \parr (\Diamond B), \tag{2}$$

but (as we shall argue) this turns out to be not as interesting as

$$\Diamond(A \parr B) \vdash (\Diamond A) \parr B; \tag{3}$$

we shall, then, consider modalities satisfying these. In fact, (3) has three other formulations, namely

$$(\Diamond A) \otimes B \vdash \Diamond(A \otimes B) \tag{4}$$

$$\Box(A \parr B) \vdash (\Box A) \parr B \tag{5}$$

$$(\Box A) \otimes B \vdash \Box(A \otimes B) \tag{6}$$

and these turn out to be equivalent:

Lemma If $\Diamond$ is functorial, and if we define $\Box A$ to be $(\Diamond A^\perp)^\perp$, then, in the presence of cut, the sequents (3), (4), (5) and (6) (with A and B free) are all equivalent.

Proof (3) and (6) can be transformed into each other by replacing A and B by there inverses, negating both sides of the entailments, and replacing $\Box$ by $\Diamond$, and similarly for (5) and (4).

From (3) we can obtain

$$\Diamond(A \parr B),\ B^\perp \vdash \Diamond A$$

by transposition. Now we replace A by $A' \otimes B'$ and B by ${B'}^\perp$, and we get

$$\Diamond((A' \otimes B') \parr {B'}^\perp),\ B' \vdash \Diamond(A' \otimes B')$$

Now trivially we have $A' \vdash (A' \otimes B') \parr {B'}^\perp$, so, by functoriality and cut, we have (4). The final equivalence is similar.

So we have several candidates for our axiom; of these, (4) is probably the most convenient.

We *need* entailments of the form (4) (rather than (2)) because rewrites have certain properties which we want to capture; in particular, if $A \rightsquigarrow B$, then, for any X, $A \otimes X \rightsquigarrow B \otimes X$. Now $A \rightsquigarrow B$ is supposed to correspond to the validity of $A \vdash \Diamond B$, and so we want the rule

$$\frac{A \vdash \Diamond B}{A \otimes X \vdash \Diamond(B \otimes X)}$$

to be admissible. A good way of ensuring this would be to have $(\Diamond B) \otimes X \vdash \Diamond(B \otimes X)$, which is just (4).

Remark 1 The comparison with the classical case is instructive here. We could just as well define modalities using rules like (4) in the classical case, but – if we have $\vee$ and $\perp$ – they would turn out to be trivial; we would find that, for any P, $\Diamond P$ was equivalent to $(\Diamond \perp) \vee P$, and $\Box P$ was equivalent to $(\Box \top) \wedge P$. We would not have added to the expressive powers of our language except by naming two arbitrary propositions $\Box\top$ and $\Diamond \perp$. In the linear case, however, we do not have these equivalences; $\Diamond P \vdash (\Diamond \perp) \parr P$ is valid, but not $(\Diamond \perp) \parr P \vdash \Diamond P$, and similarly $(\Box\mathbf{1}) \otimes P \vdash \Box P$ is valid, but not $\Box P \vdash (\Box\mathbf{1}) \otimes P$.

Intuitionistically, however, things are more interesting. In a cartesian closed category – whose internal logic is, of course, intuitionistic – we can define non-trivial $\wedge$-strong monads, and there are many interesting such. The theory of these is involved in the semantics of partial functions. [17, 19, 18]

In sequent calculus, these issues show themselves in the left rule for $\Diamond$ (and also, of course, the right rule for $\Box$). The classical rule is

$$\frac{\Box\Gamma, A \vdash \Diamond\Delta}{\Box\Gamma, \Diamond A \vdash \Diamond\Delta}$$

If we adopt (4), we get the following left rule:

$$\frac{\Gamma, A \vdash \Diamond B, \Delta}{\Gamma, \Diamond A \vdash \Diamond B, \Delta.} \Diamond\mathrm{L}$$

That is, in order to apply the left rule for $\Diamond$, we need *exactly one* modalised formula $\Diamond B$ on the right (or also a $\Box B$ on the left). This seems to make sense: examples indicate that it is reasonable to have

$$\Diamond\sigma_1',\ \alpha' \ \vdash\ \Diamond\sigma_2'$$

whenever we have

$$\sigma_1',\ \alpha' \ \vdash\ \Diamond\sigma_2',$$

but that neither

$$\Diamond\sigma_1',\ \Box\alpha' \ \vdash\ \Diamond\sigma_2',$$

nor

$$\sigma_1',\ \Box\alpha' \ \vdash\ \Diamond\sigma_2',$$

have much physical significance in the cases we are considering.

So we are led to consider modalities given by the rules in Table 2; we will call these *strong* modalities (the category-theoretic counterpart of (4) is called a *strength*). Conversely, the usual S4 rules (Table 1) will be called *monoidal* modalities (since (2) makes $\Diamond$ a monoid in the appropriate category of endofunctors).

This system, based on classical linear logic together with a strong modality, will be called $\mathsf{LL}_\Diamond$; it will be our point of departure.

Table 1 Sequent Calculus Rules for S4

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \Box A \vdash \Delta}\,\Box\mathrm{L} \qquad \frac{\Box\Gamma \vdash B, \Diamond\Delta}{\Box\Gamma \vdash \Box B, \Diamond\Delta}\,\Box\mathrm{R}$$

$$\frac{\Box\Gamma, A \vdash \Diamond\Delta}{\Box\Gamma, \Diamond A \vdash \Diamond\Delta}\,\Diamond\mathrm{L} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \Diamond A, \Delta}\,\Diamond\mathrm{R}$$

Table 2 The System $\mathsf{LL}_\Diamond$

$$\frac{\Gamma,\ A \vdash \Delta}{\Gamma,\ \Box A \vdash \Delta}\,\Box\mathrm{L} \qquad \frac{\Gamma,\ \Box A \vdash B, \Delta}{\Gamma,\ \Box A \vdash \Box B,\ \Delta}\,\Box\mathrm{R}_1 \qquad \frac{\Gamma \vdash \Diamond A,\ B,\ \Delta}{\Gamma \vdash \Diamond A,\ \Box B,\ \Delta}\,\Box\mathrm{R}_2$$

$$\frac{\Gamma,\ A \vdash \Diamond B,\ \Delta}{\Gamma,\ \Diamond A \vdash \Diamond B,\ \Delta}\,\Diamond\mathrm{L}_1 \qquad \frac{\Gamma,\ A,\ \Box B \vdash \Delta}{\Gamma,\ \Diamond A,\ \Box B \vdash \Delta}\,\Diamond\mathrm{L}_2 \qquad \frac{\Gamma \vdash A,\ \Delta}{\Gamma \vdash \Diamond A,\ \Delta}\,\Diamond\mathrm{R}$$

Table 3 The System $\mathsf{LL}_\Diamond{}'$

$$\frac{A \vdash \Diamond B,\ \Delta}{\Diamond A \vdash \Diamond B,\ \Delta}\,\Diamond\mathrm{L} \qquad \frac{\Gamma \vdash A,\ \Delta}{\Gamma \vdash \Diamond A,\ \Delta}\,\Diamond\mathrm{R}$$

$$\frac{}{\Box A \vdash (\Diamond A^\perp)^\perp} \qquad \frac{}{(\Diamond A^\perp)^\perp \vdash \Box A}$$

1.2.2 Axioms

As we have said, we intend to represent the possibility of a ramification transition, from state s_1 to state s_2, by the provability of a sequent $s_1 \vdash \Diamond s_2$. Now we have talked about a logic, but we also need to write down axioms for stipulating which transitions are possible. So, if we have an allowable transition $\sigma \rightsquigarrow \sigma'$ (where σ and σ' represent states), we want the validity of

$$\sigma \vdash \Diamond \sigma'.$$

A good way of guaranteeing this is to have a rule

$$\frac{\Gamma \vdash A,\, \Delta}{\Gamma \vdash \Diamond(A'),\, \Delta}\,\Diamond\mathrm{R}$$

whenever $A \rightsquigarrow A'$.

This gives us a right rule for $\Diamond(\sigma')$ in the presence of rewrites; we would like a corresponding left rules. There turn out to be two choices here: they are given in Tables 5 and 6. There seem to be no *phenomenological* grounds to prefer one set of axioms to the other; however, there are good technical reasons preferring the system given by Table 6. Both systems satisfy cut elimination, but only the system of Table 6 is functorial (that is, only in this system do we have $\Diamond A \vdash \Diamond B$ whenever we have $A \vdash B$).

1.2.3 Detecting Termination: the Subcategory of States

There is, finally, a further complication, which is that we cannot, merely with this machinery, detect termination of the sequence of ramification transitions: $A \vdash \Diamond B$ is valid if B can be reached from A by *any* sequence of transitions, rather than by one of maximal length. We have, then, no notion of what Sandewall calls *stable* states.

Indeed, we have no notion of states *in the logic*. We *do* use the idea of states in the metatheory; we have assumed that the basic rewrite relation leads from states to states, and that states are closed under $\otimes$. However, we still cannot assert, within the logic, that a particular proposition represents a state. So we introduce a propositional operator, $\boldsymbol{\sigma}(\cdot)$; $\boldsymbol{\sigma}(A)$ will be the state described by the proposition A. Furthermore, we will require our states to be closed under $\oplus$ as well as under $\otimes$; this will give us disjunctive states, which will allow us somewhat more flexibility when translating between representations.

So, first, we define a syntactic notion of *state proposition*; these will be the formulae that can appear inside the operator $\boldsymbol{\sigma}(\cdot)$.

Definition 1 (State Propositions) We suppose that we are given a collection of ground atomic formulae, the *state atoms*. The *state propositions* are generated from the state atoms by $\otimes$ and $\oplus$.

Note that, because $A \otimes (B \oplus C) \dashv\vdash (A \otimes B) \oplus (A \otimes C)$, state propositions have a normal form ($\oplus$s of $\otimes$s of state atoms), and that the validity of entailments between state propositions is, consequently, decidable.

Let us now assume that our rewrite relation, $\rightsquigarrow$, holds between tensor products of state atoms; we can extend it to state propositions in general as follows:

Definition 2 The rewrite relation on state propositions is defined as follows:

Table 4 Rules for $\boldsymbol{\sigma}(\cdot)$ and $\boldsymbol{\tau}(\cdot)$

$$\frac{\Gamma,\, A \vdash \Delta}{\Gamma,\, \boldsymbol{\sigma}(A) \vdash \Delta}\,\boldsymbol{\sigma}(\cdot)\mathrm{L}^{1} \qquad \frac{\Gamma \vdash A,\, \Delta}{\Gamma \vdash \boldsymbol{\sigma}(A),\, \Delta}\,\boldsymbol{\sigma}(\cdot)\mathrm{R}^{2}$$

$$\frac{\Gamma,\, \boldsymbol{\sigma}(A),\, \boldsymbol{\sigma}(A') \vdash \Delta}{\Gamma,\, \boldsymbol{\sigma}(A \otimes A') \vdash \Delta}\,\boldsymbol{\sigma}(\otimes)\mathrm{L} \qquad \frac{\Gamma \vdash \boldsymbol{\sigma}(A),\, \Delta \quad \Gamma' \vdash \boldsymbol{\sigma}(A'),\, \Delta'}{\Gamma,\, \Gamma' \vdash \boldsymbol{\sigma}(A \otimes A'),\, \Delta,\, \Delta'}\,\boldsymbol{\sigma}(\otimes)\mathrm{R}$$

$$\frac{\Gamma,\, \boldsymbol{\sigma}(A) \vdash \Delta \quad \Gamma,\, \boldsymbol{\sigma}(A') \vdash \Delta}{\Gamma,\, \boldsymbol{\sigma}(A \oplus A') \vdash \Delta}\,\boldsymbol{\sigma}(\oplus)\mathrm{L} \qquad \frac{\Gamma \vdash \boldsymbol{\sigma}(A),\, \Delta}{\Gamma \vdash \boldsymbol{\sigma}(A \oplus A'),\, \Delta}\,\boldsymbol{\sigma}(\oplus)\mathrm{R}^{3}$$

$$\frac{\Gamma,\, \boldsymbol{\sigma}(A_1) \vdash \Delta}{\Gamma,\, \boldsymbol{\tau}(A) \vdash \Delta}\,\boldsymbol{\tau}(\rightsquigarrow)\mathrm{L}^{4} \qquad \frac{\{\Gamma \vdash \boldsymbol{\sigma}(A_i),\, \Delta\}_{\boldsymbol{\sigma}(A) \rightsquigarrow \boldsymbol{\sigma}(A_i)}}{\Gamma \vdash \boldsymbol{\tau}(A),\, \Delta}\,\boldsymbol{\tau}(\rightsquigarrow)\mathrm{R}$$

$$\frac{\Gamma,\, \boldsymbol{\tau}(A) \otimes \boldsymbol{\tau}(B) \vdash \Delta}{\Gamma,\, \boldsymbol{\tau}(A \otimes B) \vdash \Delta}\,\boldsymbol{\tau}(\otimes)\mathrm{L} \qquad \frac{\{\Gamma \vdash \boldsymbol{\sigma}(A_i) \otimes B,\, \Delta\}_{\boldsymbol{\sigma}(A) \rightsquigarrow \boldsymbol{\sigma}(A_i)}}{\Gamma \vdash \boldsymbol{\tau}(A) \otimes B,\, \Delta}\,\boldsymbol{\tau}(\rightsquigarrow) \otimes \mathrm{R}$$

$$\frac{\Gamma,\, \boldsymbol{\tau}(A) \vdash \Delta \quad \Gamma,\, \boldsymbol{\tau}(B) \vdash \Delta}{\Gamma,\, \boldsymbol{\tau}(A \oplus B) \vdash \Delta}\,\boldsymbol{\tau}(\oplus)\mathrm{L}$$

[1] A a state proposition.
[2] A a state proposition.
[3] A' a state formula.
[4] $\boldsymbol{\sigma}(A) \rightsquigarrow \boldsymbol{\sigma}(A_1)$

- $\rightsquigarrow$ coincides with the original rewrite relation on tensor products of atoms;
- If $A \rightsquigarrow B$, then $A \rightsquigarrow B \oplus B'$, for B' a state proposition;
- If $A \rightsquigarrow B$ and $A' \rightsquigarrow B$, then $A \oplus A' \rightsquigarrow B$;
- $\rightsquigarrow$ is transitive.

Provided that our original rewrite relation is decidable, this extension will be also.

We can now introduce our two operators, $\sigma(\cdot)$ and $\boldsymbol{\tau}(\cdot)$; $\sigma(A)$ will mean that A is a state proposition, and the rules governing $\boldsymbol{\tau}$ are such that $\sigma(A) \vdash \boldsymbol{\tau}(A)$ is valid precisely when A is terminal. These two operators are given by the rules in Table 4.

Remark 2 We need to introduce the operator $\sigma(\cdot)$ for the following reason. In order for the rules $\boldsymbol{\tau}(\rightsquigarrow)\mathrm{R}$ and $\boldsymbol{\tau}(\rightsquigarrow) \otimes \mathrm{R}$ to be acceptible as logical rules, their antecedents should be decidable sets; so the $\rightsquigarrow$ relation, on whatever domain it is defined, should be decidable. On the other hand, it is technically advisable that $\rightsquigarrow$ should be compatible with logical inference on the domain on which it is defined (and, in particular, if we have $X \rightsquigarrow Y$, and if Y is logically equivalent to Y', we should have $X \rightsquigarrow Y'$). Defining $\rightsquigarrow$ over the state propositions satisfies both of these requirements; $\sigma(\cdot)$ is essentially a sort of state propositions.

So, finally, we can construct a logical system in which the successful execution of an action α in a situation s, leading, after ramification, to a stable situation s', corresponds to the validity of a sequent

$$\boldsymbol{\sigma}(s),\, \alpha \vdash \Diamond(\boldsymbol{\tau}(s')).$$

As we have said, the fact that these systems satisfy cut elimination (or very nearly) means that they satisfy an appropriately modified subformula property; this means that they are good candidates for a computational interpretation, using proof search to compute with actions. Formally, we define it as follows:

Definition 3 ($\mathrm{LL}^{\leadsto}_{\Diamond,\boldsymbol{\sigma},\boldsymbol{\tau}}$) The system $\mathsf{LL}^{\leadsto}_{\Diamond,\boldsymbol{\sigma},\boldsymbol{\tau}}$ is $\mathsf{LL}^{\leadsto}_{\Diamond}1$ together with the operators $\boldsymbol{\sigma}(\cdot)$ and $\boldsymbol{\tau}(\cdot)$, satisfying the rules in Table 4; rewrites now hold between propositions of the form $\sigma(A)$, according to Definition 2.

2 Cut Elimination

2.1 Cut Elimination with $\Diamond$

We will now start on the cut elimination results for our various systems. First we consider the system $\mathsf{LL}_{\Diamond}$, that is linear logic (without exponentials) augmented by the rules in Table 2.

Theorem (Cut Elimination for $\mathrm{LL}_{\Diamond}$) $\mathsf{LL}_{\Diamond}$ satisfies cut elimination.

Proof We prove this by the usual induction. Suppose that we have a cut of the form

$$\dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma \vdash C,\, \Delta\end{matrix} \qquad \begin{matrix}\Pi'\\ \vdots\\ \Gamma',\, C \vdash \Delta'\end{matrix}}{\Gamma,\, \Gamma' \vdash \Delta,\, \Delta'}\,\mathrm{cut}$$

There are several cases.

Axiom One of the sequents is an axiom, say the right one. We have, then,

$$\dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma \vdash C,\, \Delta\end{matrix} \qquad \dfrac{}{C \vdash C}}{\Gamma,\, \Gamma' \vdash \Delta,\, C}\,\mathrm{cut}$$

and the cut is essentially redundant.

Non-Principal The cut formula is non-principal in at least one of the sequents; say the right one. We now proceed by induction on the type of the principal formula in that sequent.

$\parr$**R** In this case, we have

$$\dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma \vdash C,\, \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\Pi'\\ \vdots\\ \Gamma',\, C \vdash A,\, B,\, \Delta'\end{matrix}}{\Gamma',\, C \vdash A \parr B,\, \Delta'}\,\parr\mathrm{R}}{\Gamma,\, \Gamma' \vdash A \parr B,\, \Delta,\, \Delta'}\,\mathrm{cut}$$

and we move the cut up, thus:

$$\dfrac{\dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma \vdash C,\ \Delta\end{matrix} \qquad \begin{matrix}\Pi'\\ \vdots\\ \Gamma',\ C \vdash A,\ B,\ \Delta'\end{matrix}}{\Gamma,\ \Gamma' \vdash A,\ B,\ \Delta,\ \Delta'}\,\text{cut}}{\Gamma,\ \Gamma' \vdash A \parr B,\ \Delta,\ \Delta'}\,\parr\text{R}$$

$\otimes$**L** Like $\parr$R.

$\multimap$ **R** Like $\parr$R.

$\otimes$**R** We have

$$\dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma \vdash C,\ \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\Pi'\\ \vdots\\ \Gamma',\ C \vdash A,\ \Delta'\end{matrix} \qquad \begin{matrix}\Pi''\\ \vdots\\ \Gamma'' \vdash B,\ \Delta''\end{matrix}}{\Gamma',\ \Gamma'',\ C \vdash A \otimes B,\ \Delta', \Delta''}\,\otimes\text{R}}{\Gamma,\ \Gamma',\ \Gamma'' \vdash A \otimes B,\ \Delta,\ \Delta', \Delta''}\,\text{cut}$$

which we transform to

$$\dfrac{\dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma \vdash C,\ \Delta\end{matrix} \qquad \begin{matrix}\Pi'\\ \vdots\\ \Gamma',\ C \vdash A,\ \Delta'\end{matrix}}{\Gamma,\ \Gamma' \vdash A,\ \Delta,\ \Delta'}\,\text{cut} \qquad \begin{matrix}\Pi''\\ \vdots\\ \Gamma'' \vdash B,\ \Delta''\end{matrix}}{\Gamma,\ \Gamma',\ \Gamma'' \vdash A \otimes B,\ \Delta,\ \Delta', \Delta''}\,\otimes\text{R}$$

$\parr$**L** Like $\otimes$R.

$\multimap$ **L** Like $\otimes$R.

$\&$**R** We start with

$$\dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma \vdash C,\ \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\Pi'\\ \vdots\\ \Gamma',\ C \vdash A,\ \Delta'\end{matrix} \qquad \begin{matrix}\Pi''\\ \vdots\\ \Gamma',\ C \vdash B,\ \Delta'\end{matrix}}{\Gamma',\ C \vdash A \& B,\ \Delta'}\,\&\text{R}}{\Gamma, \Gamma' \vdash A \& B,\ \Delta,\ \Delta'}\,\text{cut}$$

and transform this to

$$\dfrac{\dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma \vdash C, \Delta\end{matrix} \qquad \begin{matrix}\Pi'\\ \vdots\\ \Gamma',\ C \vdash A,\ \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash A,\ \Delta,\ \Delta'} \qquad \dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma \vdash C,\ \Delta\end{matrix} \qquad \begin{matrix}\Pi''\\ \vdots\\ \Gamma',\ C \vdash B,\ \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash B,\ \Delta,\ \Delta'}}{\Gamma, \Gamma' \vdash A \& B,\ \Delta,\ \Delta'}\,\&\text{R}$$

$\oplus$**L** Like $\&$R.

$\forall$**R** Like &R.

$\exists$**L** Like &R.

$\oplus$**R**$_1$ We have

$$\dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma \vdash C,\ \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\Pi'\\ \vdots\\ \Gamma', C \vdash A,\ \Delta\end{matrix}}{\Gamma', C \vdash A \oplus B,\ \Delta}\,\oplus\mathrm{R}_1}{\Gamma,\, \Gamma' \vdash A \oplus B,\ \Delta, \Delta'}\,\mathrm{cut}$$

and pushing the cut upwards we get

$$\dfrac{\dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma \vdash C,\ \Delta\end{matrix} \qquad \begin{matrix}\Pi'\\ \vdots\\ \Gamma', C \vdash A,\ \Delta'\end{matrix}}{\Gamma,\, \Gamma' \vdash A,\ \Delta,\ \Delta'}\,\mathrm{cut}}{\Gamma,\, \Gamma' \vdash A \oplus B,\ \Delta,\ \Delta'}\,\oplus\mathrm{R}_1$$

$\oplus$**R**$_2$ Like $\oplus$R$_1$.

&**L**$_1$ Like $\oplus$R$_1$.

&**L**$_2$ Like $\oplus$R$_1$.

$\exists$**R** Like $\oplus$R$_1$.

$\forall$**L** Like $\oplus$R$_1$.

$\Diamond$**R** We start with

$$\dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma \vdash C,\ \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\Pi'\\ \vdots\\ \Gamma',\ C \vdash A,\ \Delta'\end{matrix}}{\Gamma',\ C \vdash \Diamond A,\ \Delta'}\,\Diamond\mathrm{R}}{\Gamma,\, \Gamma' \vdash \Diamond A,\ \Delta, \Delta'}\,\mathrm{cut}$$

and we can transform this to

$$\dfrac{\dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma \vdash C,\ \Delta\end{matrix} \qquad \begin{matrix}\Pi'\\ \vdots\\ \Gamma',\ C \vdash A,\ \Delta\end{matrix}}{\Gamma,\, \Gamma' \vdash A,\ \Delta,\ \Delta'}\,\mathrm{cut}}{\Gamma,\, \Gamma' \vdash \Diamond A,\ \Delta,\ \Delta'}\,\Diamond\mathrm{R}$$

$\Box$**L** Like $\Diamond$R.

□R$_1$ There are two cases here, because this rule has a side formula, which can either be the cut formula or not. The subcase where it is not the cut formula is straightforward: we have

$$\dfrac{\dfrac{\Pi}{\Gamma \vdash C,\ \Delta} \qquad \dfrac{\dfrac{\Pi'}{\Gamma',\ \Box A,\ C \vdash B,\ \Delta'}}{\Gamma',\ \Box A,\ C \vdash \Box B,\ \Delta'}\ \Box\mathrm{R}_1}{\Gamma,\ \Gamma',\ \Box A \vdash \Box B,\ \Delta,\ \Delta'}\ \mathrm{cut}$$

and we can derive from this

$$\dfrac{\dfrac{\dfrac{\Pi}{\Gamma \vdash C,\ \Delta} \qquad \dfrac{\Pi'}{\Gamma',\ C,\ \Box A \vdash B,\ \Delta'}}{\Gamma,\ \Gamma',\ \Box A \vdash B,\ \Delta,\ \Delta'}\ \mathrm{cut}}{\Gamma,\ \Gamma',\ \Box A \vdash \Box B,\ \Delta,\ \Delta'}\ \Box\mathrm{R}_1$$

If, on the other hand, the cut formula is the side formula of this application of □R$_1$, we must have

$$\dfrac{\dfrac{\Pi}{\Gamma \vdash \Box C,\ \Delta} \qquad \dfrac{\dfrac{\Pi'}{\Gamma',\ \Box C \vdash B,\ \Delta'}}{\Gamma',\ \Box C \vdash \Box B,\ \Delta'}\ \Box\mathrm{R}_1}{\Gamma,\ \Gamma' \vdash \Box B,\ \Delta,\ \Delta'}\ \mathrm{cut}$$

We now have to examine the proof Π. If the bottom inference is an axiom, or if C is not principal in the bottom inference of Π and that inference is ⅋R, ⊗L, ⊸ R, ⊗R, ⅋L, ⊸ L, &R, ⊕L, ⊕R$_1$, ⊕R$_2$, &L$_1$, &L$_2$, ◊R, or □L, we can immediately push the cut up on the left, rather than on the right, using exactly the same reasoning as above. If the bottom inference is □R$_1$, □R$_2$, ◊L$_1$, or ◊L$_2$, and if the cut formula is neither principal nor a side formula, then we can do the same. The cut formula cannot be the side formula of any of the modal rules; none of them has a necessitated side formula on the right.

So the only subcase left is where the proof is as above and the cutformula is principal. In this case, the bottom inference must be □R$_1$ or □R$_2$; suppose, for definiteness, that the inference is □R$_1$. We then have

$$\dfrac{\dfrac{\dfrac{\tilde{\Pi}}{\Gamma,\ \Box A \vdash C,\ \Delta}}{\Gamma,\ \Box A \vdash \Box C,\ \Delta}\ \Box\mathrm{R}_1 \qquad \dfrac{\dfrac{\Pi'}{\Gamma',\ \Box C \vdash B,\ \Delta'}}{\Gamma',\ \Box C \vdash \Box B,\ \Delta'}\ \Box\mathrm{R}_1}{\Gamma,\ \Gamma',\ \Box A \vdash \Box B,\ \Delta,\ \Delta'}\ \mathrm{cut}$$

which we transform to

$$\dfrac{\dfrac{\dfrac{\dfrac{\tilde{\Pi}}{\Gamma,\,\Box A \vdash C,\,\Delta}}{\Gamma,\,\Box A \vdash \Box C,\,\Delta} \qquad \dfrac{\Pi'}{\Gamma',\,\Box C \vdash B,\,\Delta'}}{\Gamma,\,\Gamma',\,\Box A \vdash B,\,\Delta,\,\Delta'}\,\text{cut}}{\Gamma,\,\Gamma',\,\Box A \vdash \Box B,\,\Delta,\,\Delta'}\,\Box\text{R}_1$$

The subcase when the bottom inference of Π is $\Box\text{R}_2$ is exactly the same.

Principal The remaining cases are when the cut formula is principal in both subproofs. We divide by cases according to the principal connective in the cut formula.

$\otimes$ We have

$$\dfrac{\dfrac{\dfrac{\Pi}{\Gamma \vdash A,\,\Delta} \qquad \dfrac{\Pi'}{\Gamma' \vdash B,\,\Delta'}}{\Gamma,\,\Gamma' \vdash A \otimes B,\,\Delta,\,\Delta'}\,\otimes\text{R} \qquad \dfrac{\dfrac{\Pi''}{\Gamma'',\,A,\,B \vdash \Delta''}}{\Gamma'',\,A \otimes B \vdash \Delta''}\,\otimes\text{L}}{\Gamma,\,\Gamma',\,\Gamma'' \vdash \Delta,\,\Delta',\,\Delta''}\,\text{cut}$$

which becomes

$$\dfrac{\dfrac{\Pi'}{\Gamma' \vdash B,\,\Delta'} \qquad \dfrac{\dfrac{\Pi}{\Gamma \vdash A,\,\Delta} \qquad \dfrac{\Pi''}{\Gamma'',\,A,\,B \vdash \Delta''}}{\Gamma,\,\Gamma'',\,B \vdash \Delta,\,\Delta''}\,\text{cut}}{\Gamma,\,\Gamma',\,\Gamma'' \vdash \Delta,\,\Delta',\,\Delta''}\,\text{cut}$$

$\wp, \multimap$ Like $\otimes$.

$\&$ The proof will either be

$$\dfrac{\dfrac{\dfrac{\Pi}{\Gamma \vdash A,\,\Delta} \qquad \dfrac{\tilde{\Pi}}{\Gamma \vdash B,\,\Delta}}{\Gamma \vdash A\&B,\,\Delta}\,\&\text{R} \qquad \dfrac{\dfrac{\Pi'}{\Gamma',\,A \vdash \Delta'}}{\Gamma',\,A\&B \vdash \Delta'}\,\&\text{L}_1}{\Gamma,\,\Gamma' \vdash \Delta,\,\Delta'}\,\text{cut}$$

or will be of the same form but with Π' ending in $\&L_2$. In the first case (the second is, of course, completely analogous) we replace the proof with

$$\dfrac{\dfrac{\Pi}{\Gamma \vdash A,\,\Delta} \qquad \dfrac{\Pi'}{\Gamma',\,A \vdash \Delta'}}{\Gamma,\,\Gamma' \vdash \Delta,\,\Delta'}\,\text{cut}$$

$\oplus$ Like &.

$\forall$ Like &.

$\exists$ Like &.

$\Box$ There are two cases, depending on the right rule applied to $\Box$ in the original proof. Suppose that it is $\Box R_1$; we have, therefore,

$$\dfrac{\dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma, \Box A \vdash B, \Delta\end{matrix}}{\Gamma, \Box A \vdash \Box B, \Delta}\,\Box R_1 \qquad \dfrac{\begin{matrix}\Pi'\\ \vdots\\ \Gamma', B \vdash \Delta'\end{matrix}}{\Gamma', \Box B \vdash \Delta'}\,\Box L}{\Gamma, \Gamma', \Box A \vdash \Delta, \Delta'}\,\text{cut}$$

and we replace this with

$$\dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma, \Box A \vdash B, \Delta\end{matrix} \qquad \begin{matrix}\Pi'\\ \vdots\\ \Gamma', B \vdash \Delta'\end{matrix}}{\Gamma, \Gamma', \Box A \vdash \Delta, \Delta'}\,\text{cut}$$

If the right rule is $\Box R_2$ the proof is, of course, much the same.

$\Diamond$ Like $\Box$.

2.2 Incorporating Rewrites as Axioms

The above cut elimination result is, of course, for systems without axioms. In practice, though, we will have in mind a particular collection of rewrites which we will want the modality to represent; so we want to know how to present systems with axioms which encode a given set of rewrites.

First we define the sort of propositions that our rewrites will act on.

DEFINITION 4 A *state proposition* will be a finite tensor product of non-modal atomic propositions.

Next we assume that we have a suitable rewrite system.

DEFINITION 5 Let $\rightsquigarrow_0$ be a set of basic rewrites: that is, it will consist of a set of pairs $\{A_i \rightsquigarrow_0 B_i \mid i \in I\}$, where A_i and B_i are state propositions for all i.

We now extend $\rightsquigarrow_0$ to the set of all state propositions by Definition 2 (p. 9).

We can now modify our rules for the modal operators to accommodate these rewrites. As before, it is simpler to give rules in the style of Table 3; that is, we first give rules for $\Diamond$, and then define $\Box$ in terms of $\Diamond$. It turns out that we can give two systems, differing in strength; the weaker one is weirdly intensional, but we will need it for representing certain things.

DEFINITION 6 Given a rewrite system $\rightsquigarrow$, we define two linear modal systems: $\mathsf{LL}_{\Diamond}^{\rightsquigarrow}0$ is given by the rules in Table 5, and $\mathsf{LL}_{\Diamond}^{\rightsquigarrow}1$ is given by the rules in Table 6.

Table 5 The System $\mathsf{LL}_{\Diamond}^{\leadsto}0$

$$\frac{\{\Gamma,\, A_i \vdash \Diamond B,\, \Delta\}_{A_i \leadsto A}}{\Gamma,\, \Diamond A \vdash \Diamond B,\, \Delta}\,\Diamond\mathrm{L} \qquad \frac{\Gamma \vdash A_1,\, \Delta}{\Gamma \vdash \Diamond A,\, \Delta}\,\Diamond\mathrm{R}$$

$$\frac{\Gamma \vdash \Diamond A^{\perp},\, \Delta}{\Gamma,\, \Box A \vdash \Delta}\,\Box\mathrm{L} \qquad \frac{\Gamma,\, \Diamond A^{\perp} \vdash \Delta}{\Gamma \vdash \Box A,\, \Delta}\,\Box\mathrm{R}$$

In $\Diamond$R, A_1 is a proposition such that $A_1 \leadsto A$; in $\Diamond$L the A_i are all of the propositions such that $A_i \leadsto A$.

Table 6 The System $\mathsf{LL}_{\Diamond}^{\leadsto}1$

$$\frac{\Gamma,\, A \vdash \Diamond B,\, \Delta}{\Gamma,\, \Diamond A \vdash \Diamond B,\, \Delta}\,\Diamond\mathrm{L} \qquad \frac{\Gamma,\, A \vdash \Diamond B,\, \Delta}{\Gamma,\, A_1,\, \ldots,\, A_r \vdash \Diamond B,\, \Delta} \leadsto \mathrm{L} \qquad \frac{\Gamma \vdash A_1,\, \Delta}{\Gamma \vdash \Diamond A,\, \Delta}\,\Diamond\mathrm{R}$$

$$\frac{\Gamma \vdash \Diamond A^{\perp},\, \Delta}{\Gamma,\, \Box A \vdash \Delta}\,\Box\mathrm{L} \qquad \frac{\Gamma,\, \Diamond A^{\perp} \vdash \Delta}{\Gamma \vdash \Box A,\, \Delta}\,\Box\mathrm{R}$$

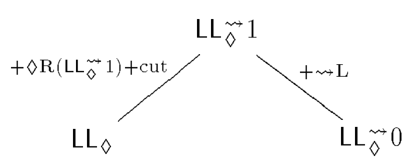
In $\Diamond$R, A_1 is a proposition such that $A_1 \leadsto A$; in $\leadsto$ L, $A_1, \ldots, A_r$ are propositions such that $(A_1 \otimes \ldots \otimes A_r) \leadsto A$.

Remark 3 We could, if we wish, modify the rule $\Diamond$L of $\mathsf{LL}_{\Diamond}^{\leadsto}0$ to

$$\frac{\{A_i \vdash \Diamond B,\, \Delta \mid A_i \leadsto_0 A, A_i \neq A\} \quad A \vdash \Diamond B,\, \Delta}{\Diamond A \vdash \Diamond B,\, \Delta}\,\Diamond\mathrm{L}'$$

and modify $\Diamond$R by only admitting A_1s with $A_1 \leadsto_0 A$. It is easy to show, using the transitivity of $\leadsto$, that the two systems are equivalent. The modified presentation would have the advantage that $\Diamond$L would have finite antecedents in many cases its first version did not. However, the rules as we have given them are more straightforward to prove cut elimination for.

Proposition The three systems $\mathsf{LL}_{\Diamond}$, $\mathsf{LL}_{\Diamond}^{\leadsto}0$, and $\mathsf{LL}_{\Diamond}^{\leadsto}1$ are related as follows:



that is, every rule of $\mathsf{LL}_{\Diamond}$ is admissible for $\mathsf{LL}_{\Diamond}^{\leadsto}1$, every rule of $\mathsf{LL}_{\Diamond}^{\leadsto}0$ is admissible for $\mathsf{LL}_{\Diamond}^{\leadsto}1$, and $\mathsf{LL}_{\Diamond}^{\leadsto}1$is obtained by adding $\Diamond\mathrm{R}(\mathsf{LL}_{\Diamond}^{\leadsto}1)$ to $\mathsf{LL}_{\Diamond}$ (in the presence of cut), or by adding $\leadsto$ L to $\mathsf{LL}_{\Diamond}^{\leadsto}0$.

Proof The admissibility part is trivial.

Suppose now that we consider the system $\mathsf{LL}_{\Diamond} + \Diamond\mathrm{R}(\mathsf{LL}_{\Diamond}^{\leadsto}1)$; this is clearly closed under all of the rules of $\mathsf{LL}_{\Diamond}^{\leadsto}1$ except maybe $\leadsto$ L. So if we have a proof

Π, in $\mathsf{LL}_\Diamond + \Diamond\mathrm{R}(\mathsf{LL}_\Diamond^{\leadsto}1)$, of the antecedent of $\leadsto$ L, we can prove its consequent as follows:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A_1, \ldots, A_r \vdash A_1 \otimes \ldots \otimes A_r\end{matrix}}{A_1, \ldots, A_r \vdash \Diamond A}\Diamond\mathrm{R}(\mathsf{LL}_\Diamond^{\leadsto}1) \qquad \dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma, A \vdash \Diamond B, \Delta\end{matrix}}{\Gamma, \Diamond A \vdash \Diamond B, \Delta}\Diamond\mathrm{L}(\mathsf{LL}_\Diamond)}{\Gamma, A_1, \ldots, A_r \vdash \Diamond B, \Delta}\,\mathrm{cut}$$

Similarly, the system $\mathsf{LL}_\Diamond^{\leadsto}0 + \leadsto$ L is clearly closed under all of the rules of $\mathsf{LL}_\Diamond^{\leadsto}1$ except possibly $\Diamond\mathrm{L}(\mathsf{LL}_\Diamond^{\leadsto}1)$. So if we have a proof Π of the antecedent of $\Diamond\mathrm{L}\mathsf{LL}_\Diamond^{\leadsto}1$, we can prove its consequent in $\mathsf{LL}_\Diamond^{\leadsto}0 + \leadsto$ L:

$$\dfrac{\left\{\dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma, A \vdash \Diamond B, \Delta\end{matrix}}{\Gamma, A_i \vdash \Diamond B, \Delta}\leadsto \mathrm{L}\right\}_{A_i \leadsto A}}{\Gamma, \Diamond A \vdash \Diamond B, \Delta}\Diamond\mathrm{L}(\mathsf{LL}_\Diamond^{\leadsto}0)$$

Lemma In $\mathsf{LL}_\Diamond^{\leadsto}0$,

$$\Gamma, \Diamond A \vdash \Diamond((\bigotimes_{\gamma \in \Gamma} \gamma) \otimes A)$$

Proof Obvious.

Lemma In $\mathsf{LL}_\Diamond^{\leadsto}0$,

$$\Diamond\Diamond B \vdash \Diamond B$$

Proof Obvious.

Proposition In the presence of cut elimination, $\mathsf{LL}_\Diamond^{\leadsto}0$ together with functoriality:

$$\frac{A \vdash B}{\Diamond A \vdash \Diamond B}$$

is equivalent to $\mathsf{LL}_\Diamond^{\leadsto}1$.

Proof We will show that, given $\mathsf{LL}_\Diamond^{\leadsto}0$ together with cut and functoriality, we can get $\leadsto$ L. Suppose, then, that $A' \leadsto A$ and that we have a proof Π of $\Gamma, A \vdash \Diamond B$. The resulting proof of $\Gamma, A' \vdash \Diamond B$ is given in Table 7.

Remark 4 This is basically rather bad news for $\mathsf{LL}_\Diamond^{\leadsto}0$; we cannot do much without functoriality of $\Diamond$. For example, we cannot automatically substitute logically equivalent propositions inside $\Diamond$. But if we impose functoriality, $\mathsf{LL}_\Diamond^{\leadsto}0$ turns into $\mathsf{LL}_\Diamond^{\leadsto}1$. Nevertheless, we will find it convenient to use $\mathsf{LL}_\Diamond^{\leadsto}0$ in its pure state occasionally; for example, as Corollary 2 shows, $\mathsf{LL}_\Diamond^{\leadsto}0$ represents Thielscher's approach to ramification fairly exactly.

We can now prove cut elimination for these systems.

Theorem (Cut Elimination for $\mathsf{LL}_\Diamond^{\leadsto}0$) Any proof Π of a sequent $\Gamma \vdash \Delta$ in the system $\mathsf{LL}_\Diamond^{\leadsto}0$ can be transformed into a proof Π' which is cut free.

Proof We need to check the following cases, which differ from the proof of Theorem 1.

Table 7 Functoriality + $\mathsf{LL}_{\Diamond}^{\rightsquigarrow}0 = \mathsf{LL}_{\Diamond}^{\rightsquigarrow}1$

$$
\dfrac{
\dfrac{\overline{A' \vdash A'}}{A' \vdash \Diamond A}\,\Diamond\mathrm{R}(\mathsf{LL}_{\Diamond}^{\rightsquigarrow}0)
\qquad
\dfrac{
\begin{array}{c}\text{Lemma 2}\\ \vdots \\ \Gamma, \Diamond A \vdash \Diamond((\bigotimes_{\gamma\in\Gamma}\gamma)\otimes A)\end{array}
\qquad
\dfrac{
\dfrac{\dfrac{\begin{array}{c}\Pi\\ \vdots\\ \Gamma, A \vdash \Diamond B\end{array}}{(\bigotimes_{\gamma\in\Gamma}\gamma)\otimes A \vdash \Diamond B}}{\Diamond((\bigotimes_{\gamma\in\Gamma}\gamma)\otimes A) \vdash \Diamond\Diamond B}\,\text{functoriality}
\qquad
\begin{array}{c}\text{Lemma 3}\\ \vdots\\ \Diamond\Diamond B \vdash \Diamond B\end{array}
}{\Diamond((\bigotimes_{\gamma\in\Gamma}\gamma)\otimes A) \vdash \Diamond B}\,\text{cut}
}{\Gamma, \Diamond A \vdash \Diamond B}\,\text{cut}
}{\Gamma, A' \vdash \Diamond B}\,\text{cut}
$$

Non-Principal Here we need to check the case where the cut is against the side formula of $\Diamond$L; we can assume that the cutformula is principal in the other subproof. Consequently, the bottom inference of the other subproof must be $\Diamond$L, and the proof looks like

$$\dfrac{\dfrac{\left\{\begin{array}{c}\Pi_i \\ \vdots \\ \Gamma,\, A_i \vdash \Diamond B,\, \Delta\end{array}\right\}_{A_i \leadsto A}}{\Gamma,\, \Diamond A \vdash \Diamond B,\, \Delta}\,\Diamond\mathrm{L} \qquad \begin{array}{c}\Pi' \\ \vdots \\ \Gamma',\, \Diamond B \vdash \Diamond C,\, \Delta'\end{array}}{\Gamma,\, \Gamma',\, \Diamond A \vdash \Diamond C,\, \Delta,\, \Delta'}\,\mathrm{cut}$$

We can move the cut upwards, which gives us this proof:

$$\dfrac{\left\{\dfrac{\begin{array}{c}\Pi_i \\ \vdots \\ \Gamma,\, A_i \vdash \Diamond B,\, \Delta\end{array} \qquad \begin{array}{c}\Pi' \\ \vdots \\ \Gamma',\, \Diamond B \vdash \Diamond C,\, \Delta'\end{array}}{\Gamma,\, \Gamma',\, A_i \vdash \Diamond C,\, \Delta,\, \Delta'}\,\mathrm{cut}\right\}_{A_i \leadsto A}}{\Gamma,\, \Gamma',\, \Diamond A \vdash \Diamond C,\, \Delta,\, \Delta'}$$

Principal We need to check the case where the cutformula is principal in $\Diamond$L and $\Diamond$R. So the proof is

$$\dfrac{\dfrac{\begin{array}{c}\Pi \\ \vdots \\ \Gamma \vdash A_j,\, \Delta\end{array}}{\Gamma \vdash \Diamond A,\, \Delta}\,\Diamond\mathrm{R} \qquad \dfrac{\left\{\begin{array}{c}\Pi_i \\ \vdots \\ \Gamma',\, A_i \vdash \Diamond B,\, \Delta'\end{array}\right\}_{A_i \leadsto A}}{\Gamma',\, \Diamond A \vdash \Diamond B,\, \Delta'}\,\Diamond\mathrm{L}}{\Gamma,\, \Gamma' \vdash \Diamond B,\, \Delta,\, \Delta'}\,\mathrm{cut}$$

Here $A_j \leadsto A$, so A_j must be one of the A_i. Consequently, we can move the cut upwards:

$$\dfrac{\begin{array}{c}\Pi \\ \vdots \\ \Gamma \vdash A_j,\, \Delta\end{array} \qquad \begin{array}{c}\Pi_j \\ \vdots \\ \Gamma',\, A_j \vdash \Diamond B,\, \Delta'\end{array}}{\Gamma,\, \Gamma' \vdash \Diamond B,\, \Delta,\, \Delta'}\,\mathrm{cut}$$

Theorem (Cut Elimination for $\mathrm{LL}_{\Diamond}^{\leadsto}1$) Any proof Π of a sequent $\Gamma \vdash \Delta$ in the system $\mathsf{LL}_{\Diamond}^{\leadsto}1$ can be transformed into a proof Π' which is cut free.

Proof Again we need to check the cases which differ from the proof of Theorem 1.

Non-Principal The new rules come into play when the cutformula is the side formula in $\Diamond$L or in $\leadsto$ L. In both cases we can assume that the cutformula is principal in the other sequent.

$\Diamond$L The proof has the form

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ \vdots\\ \Gamma,\, A \vdash \Diamond B,\, \Delta\end{array}}{\Gamma,\, \Diamond A \vdash \Diamond B,\, \Delta}\,\Diamond\text{L} \qquad \begin{array}{c}\Pi'\\ \vdots\\ \Gamma',\, \Diamond B \vdash \Diamond C,\, \Delta'\end{array}}{\Gamma,\, \Gamma',\, \Diamond A \vdash \Diamond C,\, \Delta,\, \Delta'}\,\text{cut}$$

and we can transform this to

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ \vdots\\ \Gamma,\, A \vdash \Diamond B,\, \Delta\end{array} \qquad \begin{array}{c}\Pi'\\ \vdots\\ \Gamma',\, \Diamond B \vdash \Diamond C,\, \Delta'\end{array}}{\Gamma,\, \Gamma',\, A \vdash \Diamond C,\, \Delta,\, \Delta'}\,\text{cut}}{\Gamma,\, \Gamma',\, \Diamond A \vdash \Diamond C,\, \Delta,\, \Delta'}\,\Diamond\text{L}$$

thus moving the cut upwards on the left.

$\rightsquigarrow$ L In this case the proof has the form

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ \vdots\\ \Gamma,\, A \vdash \Diamond B,\, \Delta\end{array}}{\Gamma,\, A_1,\, \dots,\, A_r \vdash \Diamond B,\, \Delta}\rightsquigarrow\text{L} \qquad \begin{array}{c}\Pi'\\ \vdots\\ \Gamma',\, \Diamond B \vdash \Diamond C,\, \Delta'\end{array}}{\Gamma,\, \Gamma',\, A_1,\, \dots,\, A_r \vdash \Diamond C,\, \Delta,\, \Delta'}\,\text{cut}$$

where $(A_1,\, \dots,\, A_r) \rightsquigarrow A$. So we can move the cut upwards on the left thus:

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ \vdots\\ \Gamma,\, A \vdash \Diamond B,\, \Delta\end{array} \qquad \begin{array}{c}\Pi'\\ \vdots\\ \Gamma',\, \Diamond B \vdash \Diamond C,\, \Delta'\end{array}}{\Gamma,\, \Gamma',\, A \vdash \Diamond C,\, \Delta,\, \Delta'}\,\text{cut}}{\Gamma,\, \Gamma',\, A_1,\, \dots,\, A_r \vdash \Diamond C,\, \Delta,\, \Delta'}\rightsquigarrow\text{L}$$

Principal We now have the cases where the cutformula is principal in both premises, and where the new rules are used in at least one of the premises. So these are as follows:

$\Diamond$R,$\Diamond$L Here the proof is as follows:

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ \vdots\\ \Gamma \vdash A_1,\, \Delta\end{array}}{\Gamma \vdash \Diamond A,\, \Delta}\,\Diamond\text{R} \qquad \dfrac{\begin{array}{c}\Pi'\\ \vdots\\ \Gamma', A \vdash \Diamond B,\, \Delta'\end{array}}{\Gamma',\, \Diamond A \vdash \Diamond B,\, \Delta'}\,\Diamond\text{L}}{\Gamma,\, \Gamma' \vdash \Diamond B,\, \Delta,\, \Delta'}\,\text{cut}$$

where $A_1 \rightsquigarrow A$. We can move the cut upwards and obtain

$$\dfrac{\begin{array}{c}\Pi \\ \vdots \\ \Gamma \vdash A_1,\, \Delta\end{array} \qquad \dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \Gamma',\, A \vdash \Diamond B,\, \Delta'\end{array}}{\Gamma',\, A_1 \vdash \Diamond B,\, \Delta'} \rightsquigarrow \mathrm{L}}{\Gamma,\, \Gamma' \vdash \Diamond B,\, \Delta,\, \Delta'} \mathrm{cut}\ .$$

⊗R,⇝ L The proof must be of the form

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1 \\ \vdots \\ \Gamma \vdash A_1,\, \Delta\end{array} \qquad \begin{array}{c}\Pi_2 \\ \vdots \\ \Gamma' \vdash A_2,\, \Delta'\end{array}}{\Gamma,\, \Gamma' \vdash A_1 \otimes A_2,\, \Delta, \Delta'} \otimes\mathrm{R} \qquad \dfrac{\begin{array}{c}\Pi'' \\ \vdots \\ \Gamma'',\, A \vdash \Diamond B,\, \Delta''\end{array}}{\Gamma'',\, A_1 \otimes A_2,\, A_3,\, \ldots,\, A_r \vdash \Diamond B,\, \Delta''}}{\Gamma,\, \Gamma',\, \Gamma'',\, A_3,\, \ldots,\, A_r \vdash \Diamond B,\, \Delta,\, \Delta',\, \Delta''} \mathrm{cut}$$

which we can replace by two cuts higher up, thus:

$$\dfrac{\begin{array}{c}\Pi_2 \\ \vdots \\ \Gamma' \vdash A_2,\, \Delta'\end{array} \qquad \dfrac{\begin{array}{c}\Pi_1 \\ \vdots \\ \Gamma \vdash A_1,\, \Delta\end{array} \qquad \dfrac{\begin{array}{c}\Pi'' \\ \vdots \\ \Gamma'',\, A \vdash \Diamond B,\, \Delta''\end{array}}{\Gamma'',\, A_1,\, A_2,\, A_3,\, \ldots,\, A_r \vdash \Diamond B,\, \Delta''} \rightsquigarrow \mathrm{L}}{\Gamma, \Gamma'',\, A_2,\, A_3,\, \ldots,\, A_r \vdash \Diamond B,\, \Delta,\, \Delta''} \mathrm{cut}}{\Gamma,\, \Gamma',\, \Gamma'',\, A_3,\, \ldots,\, A_r \vdash \Diamond B,\, \Delta,\, \Delta',\, \Delta''}$$

(the side conditions for the $\rightsquigarrow$ L rule are, for both proofs, that $(A_1 \otimes \ldots \otimes A_r) \rightsquigarrow A$).

⊕R, ⇝ L The proof must look like

$$\dfrac{\dfrac{\begin{array}{c}\Pi \\ \vdots \\ \Gamma \vdash A,\, \Delta\end{array}}{\Gamma \vdash A \oplus B,\, \Delta} \qquad \dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \Gamma',\, C \vdash \Diamond D,\, \Delta'\end{array}}{\Gamma',\, A \oplus B \vdash \Diamond D,\, \Delta'}}{\Gamma,\, \Gamma' \vdash \Diamond D,\, \Delta,\, \Delta'}$$

where $A \oplus B \rightsquigarrow C$. Now if $A \rightsquigarrow A \oplus B$, so, by the transitivity of $\rightsquigarrow$, we must have $A \rightsquigarrow C$, so we can move the cut upwards thus:

$$\dfrac{\begin{array}{c}\Pi \\ \vdots \\ \Gamma \vdash A,\, \Delta\end{array} \qquad \dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \Gamma',\, C \vdash \Diamond D,\, \Delta'\end{array}}{\Gamma',\, A \vdash \Diamond D,\, \Delta'}}{\Gamma,\, \Gamma' \vdash \Diamond D,\, \Delta,\, \Delta'}$$

Finally we state and prove a theorem about the permutability of the various rules.

Theorem (Permutabilities for $\mathrm{LL}^{\leadsto}_{\Diamond}1$) The only cases in which the rules of $\mathsf{LL}^{\leadsto}_{\Diamond}1$ cannot be permuted with each other are the following:

1. Neither $\otimes$R nor ⅋L can be permuted below ⅋R or $\otimes$L.
2. Neither $\otimes$R nor ⅋L can be permuted below &R or $\oplus$L.
3. Neither $\otimes$R nor ⅋L can be permuted below $\Diamond$L, $\Box$R, or $\leadsto$ L.
4. Neither $\oplus$R nor &L can be permuted below &R or $\oplus$L.
5. Neither $\exists$R nor $\forall$L can be permuted below &R or $\oplus$L.
6. Neither $\exists$R nor $\forall$L can be permuted below $\forall$R or $\exists$L.
7. Neither $\Diamond$L nor $\Box$L can be permuted below $\Diamond$L, $\Box$R, or $\leadsto$ L.

Proof We can quote Troelstra [27, pp. 340f.] (who cites Lincoln [12]), or Pym and Harland [21], for the permutabilities involving the non-modal rules. The permutability of $\Diamond$R and $\Box$L below the non-modal rules is clear, as is the permutability of $\Diamond$L and $\Box$R below the &, $\oplus$, $\forall$ and $\exists$ rules. Finally we need counterexamples for the stated exceptions. A counterexample for the non-permutability of $\otimes$R under $\Diamond$L (3) is given by:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A} \qquad \dfrac{\dfrac{}{B \vdash B}}{\Box B \vdash B}}{A,\, \Box B \vdash A \otimes B}}{\Diamond A,\, \Box B \vdash A \otimes B}$$

The example for $\otimes$R and $\Box$R is, of course, similar. A counterexample for the non-permutability of $\otimes$R under $\leadsto$ L is given by

$$\dfrac{\dfrac{\dfrac{}{A_1 \vdash A_1} \qquad \dfrac{\dfrac{}{B \vdash B}}{\Box B \vdash B}}{A_1,\, \Box B \vdash A_1 \otimes B}}{A,\, \Box B \vdash A_1 \otimes B} \leadsto \mathrm{L}$$

where $A \leadsto A_1$.

Finally, a counterexample for 7 is given by the obvious (indeed the only) cut-free proof of $\Diamond A \vdash \Diamond A$, for A atomic.

2.3 Cut Elimination with $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$

We now give cut elimination results for the system $\mathsf{LL}^{\leadsto}_{\Diamond,\boldsymbol{\sigma},\boldsymbol{\tau}}$. First, however, a foolish little lemma:

Lemma If we have a cut-free proof of $\Gamma,\, \boldsymbol{\tau}(A) \vdash \Delta$, then there is a cut-free proof of some sequent $\Gamma,\, \boldsymbol{\sigma}(A_i) \vdash \Delta$ for some A_i with $\boldsymbol{\sigma}(A) \leadsto \boldsymbol{\sigma}(A_i)$.

Proof An obvious induction on the size of the proof.

Theorem (Cut Elimination for $\mathrm{LL}^{\leadsto}_{\Diamond,\boldsymbol{\sigma},\boldsymbol{\tau}}$) Any proof in $\mathsf{LL}^{\leadsto}_{\Diamond,\boldsymbol{\sigma},\boldsymbol{\tau}}$ can be transformed into one which has no cuts.

Proof Since we have already proved cut elimination for $\mathsf{LL}^{\leadsto}_{\Diamond}1$, there are two things to check; firstly, that cut elimination for $\mathsf{LL}^{\leadsto}_{\Diamond}1$ is not broken by the new operators, and, secondly, that we can eliminate cuts involving the new operators.

In order to show that we can still carry out cut elimination for $\mathsf{LL}^{\leadsto}_{\Diamond}1$ in the presence of the new operators, we must show that cuts can still be moved upwards across rules in which they are non-principal. But this is immediate; the new rules which we have introduced are either single premise rules or behave like &R, so that moving cuts upwards across them is straightforward. What is crucial here is that none of the $\boldsymbol{\sigma}(\cdot)$ or $\boldsymbol{\tau}(\cdot)$ rules has side formulae.

Now we must deal with the case where the cutformula is of the form $\boldsymbol{\sigma}(A)$ or $\boldsymbol{\tau}(A)$ and is principal in both premises of the cut.

Cuts with $\boldsymbol{\sigma}(A)$ First we deal with cuts involving $\boldsymbol{\sigma}(A)$. The rules which have $\boldsymbol{\sigma}(A)$ as a principal formula are the $\boldsymbol{\sigma}$ rules given in Table 4, and also the rule $\leadsto$ L.

$\boldsymbol{\sigma}(\cdot)$R, $\boldsymbol{\sigma}(\cdot)$L The cut looks like this

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash A,\ \Delta\end{array}}{\Gamma \vdash \boldsymbol{\sigma}(A),\ \Delta}\,\boldsymbol{\sigma}(\cdot)\mathrm{R} \qquad \dfrac{\begin{array}{c}\vdots\\ \Gamma',\ A \vdash \Delta'\end{array}}{\Gamma',\ \boldsymbol{\sigma}(A) \vdash \Delta'}\,\boldsymbol{\sigma}(cdot)\mathrm{L}}{\Gamma,\ \Gamma' \vdash \Delta,\ \Delta'}\,\mathrm{cut}$$

and moving it upwards is trivial.

$\boldsymbol{\sigma}(\cdot)$R, $\boldsymbol{\sigma}(\otimes)$L The cut looks like this:

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ \vdots\\ \Gamma \vdash A \otimes B,\ \Delta\end{array}}{\Gamma \vdash \boldsymbol{\sigma}(A \otimes B),\ \Delta}\,\boldsymbol{\sigma}(\cdot)\mathrm{R} \qquad \dfrac{\begin{array}{c}\Pi'\\ \vdots\\ \Gamma',\ \boldsymbol{\sigma}(A),\ \boldsymbol{\sigma}(B) \vdash \Delta'\end{array}}{\Gamma',\ \boldsymbol{\sigma}(A \otimes B) \vdash \Delta'}}{\Gamma,\ \Gamma' \vdash \Delta,\ \Delta'}\,\mathrm{cut}$$

There are two cases. If $A \otimes B$ is not principal in Π, then, since $A \otimes B$ is not modal and consequently cannot be a side formula, we can move $\boldsymbol{\sigma}(\cdot)$R (and with it the cut) upwards. Otherwise, we can write $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Delta = \Delta_1 \cup \Delta_2$, and we have proofs Π_1 of $\Gamma_1 \vdash A,\ \Delta_1$ and Π_2 of $\Gamma_2 \vdash B,\ \Delta_2$ out of which Π arises by $\otimes$R. So we can transform the proof to

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \vdots\\ \Gamma_2 \vdash B,\ \Delta_2\end{array}}{\Gamma_2 \vdash \boldsymbol{\sigma}(B),\ \Delta_2} \qquad \dfrac{\dfrac{\begin{array}{c}\Pi_2\\ \vdots\\ \Gamma_1 \vdash A,\ \Delta_1\end{array}}{\Gamma_1 \vdash \boldsymbol{\sigma}(A),\ \Delta_1} \quad \begin{array}{c}\Pi'\\ \vdots\\ \Gamma',\ \boldsymbol{\sigma}(A),\ \boldsymbol{\sigma}(B) \vdash \Delta'\end{array}}{\Gamma_1,\ \Gamma',\ \boldsymbol{\sigma}(B) \vdash \Delta_1,\ \Delta'}\,\mathrm{cut}}{\Gamma_1,\ \Gamma_2,\ \Gamma' \vdash \Delta_1,\ \Delta_2,\ \Delta'}\,\mathrm{cut}$$

which moves the cut upwards.

$\boldsymbol{\sigma}(\cdot)$**R**, $\boldsymbol{\sigma}(\oplus)$**L** Here the proof looks like

$$\dfrac{\dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma \vdash A \oplus B,\ \Delta\end{matrix}}{\Gamma \vdash \boldsymbol{\sigma}(A \oplus B),\ \Delta} \qquad \dfrac{\begin{matrix}\vdots\\ \Gamma',\ \boldsymbol{\sigma}(A) \vdash \Delta'\end{matrix} \qquad \begin{matrix}\vdots\\ \Gamma',\ \boldsymbol{\sigma}(B) \vdash \Delta'\end{matrix}}{\Gamma',\ \boldsymbol{\sigma}(A \oplus B) \vdash \Delta'}}{\Gamma,\ \Gamma' \vdash \Delta,\ \Delta'}\ \text{cut}$$

Here we argue by cases, according to whether $A \oplus B$ is principal in Π or not; the argument is much like that of the previous case.

$\boldsymbol{\sigma}(\otimes)$**R**, $\boldsymbol{\sigma}(\cdot)$**L** Here the proof is of the form

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma_1 \vdash \boldsymbol{\sigma}(A),\ \Delta_1\end{matrix} \qquad \begin{matrix}\vdots\\ \Gamma_2 \vdash \boldsymbol{\sigma}(B),\ \Delta_2\end{matrix}}{\Gamma_1,\ \Gamma_2 \vdash \boldsymbol{\sigma}(A \otimes B),\ \Delta_1,\ \Delta_2} \qquad \dfrac{\begin{matrix}\Pi'\\ \vdots\\ \Gamma',\ A \otimes B \vdash \Delta'\end{matrix}}{\Gamma',\ \boldsymbol{\sigma}(A \otimes B) \vdash \Delta'}}{\Gamma_1,\ \Gamma_2,\ \Gamma' \vdash \Delta_1,\ \Delta_2,\ \Delta'}\ \text{cut}$$

and we argue by cases on whether $A \otimes B$ is principal in Π' or not.

$\boldsymbol{\sigma}(\otimes)$**R**, $\boldsymbol{\sigma}(\otimes)$**L** We have

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma_1 \vdash \boldsymbol{\sigma}(A),\ \Delta_1\end{matrix} \qquad \begin{matrix}\vdots\\ \Gamma_2 \vdash \boldsymbol{\sigma}(B),\ \Delta_2\end{matrix}}{\Gamma_1,\ \Gamma_2 \vdash \boldsymbol{\sigma}(A \otimes B),\ \Delta_1,\ \Delta_2} \qquad \dfrac{\begin{matrix}\vdots\\ \Gamma',\ \boldsymbol{\sigma}(A),\ \boldsymbol{\sigma}(B) \vdash \Delta'\end{matrix}}{\Gamma',\ \boldsymbol{\sigma}(A \otimes B) \vdash \Delta'}}{\Gamma_1,\ \Gamma_2,\ \Gamma' \vdash \Delta_1,\ \Delta_2,\ \Delta'}\ \text{cut}$$

and moving the cut upwards is trivial.

$\boldsymbol{\sigma}(\oplus)$**R**, $\boldsymbol{\sigma}(\cdot)$**L** The proof must be of the form

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma \vdash \boldsymbol{\sigma}(A),\ \Delta\end{matrix}}{\Gamma \vdash \boldsymbol{\sigma}(A \oplus B),\ \Delta} \qquad \dfrac{\begin{matrix}\Pi'\\ \vdots\\ \Gamma',\ A \oplus B \vdash \Delta'\end{matrix}}{\Gamma',\ \boldsymbol{\sigma}(A \oplus B) \vdash \Delta'}}{\Gamma,\ \Gamma' \vdash \Delta,\ \Delta'}\ \text{cut}$$

As usual, we argue by cases according to whether $A \oplus B$ is principal in Π' or not.

$\boldsymbol{\sigma}(\oplus)$**R**, $\boldsymbol{\sigma}(\oplus)$**L** Again trivial.

$\boldsymbol{\sigma}(\cdot)$**R**, $\rightsquigarrow$ **L** We start with the proof

$$\dfrac{\dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma \vdash A,\ \Delta\end{matrix}}{\Gamma \vdash \boldsymbol{\sigma}(A),\ \Delta}\ \boldsymbol{\sigma}(\cdot)\text{R} \qquad \dfrac{\begin{matrix}\vdots\\ \Gamma',\ \boldsymbol{\sigma}(A_1) \vdash \Diamond B,\ \Delta'\end{matrix}}{\Gamma',\ \boldsymbol{\sigma}(A) \vdash \Diamond B,\ \Delta'}}{\Gamma,\ \Gamma' \vdash \Diamond B,\ \Delta,\ \Delta'}\ \text{cut}$$

where $\boldsymbol{\sigma}(A) \rightsquigarrow \boldsymbol{\sigma}(A_1)$. We argue by cases, according to whether A is principal in Π or not. If it is not, we can simply permute the rule $\boldsymbol{\sigma}(\cdot)$R, and with it the cut, upwards in Π. If A is principal, then Π is an axiom or is $\otimes$R or $\oplus$R. If Π is an axiom, we can trivially eliminate the cut; if Π starts with $\otimes$R, we can permute $\boldsymbol{\sigma}(\cdot)$R upwards and replace $\otimes$R with $\boldsymbol{\sigma}(\otimes)$R. This reduces this case to the next one. The argument for the case when Π starts with $\oplus$R is entirely similar.

$\boldsymbol{\sigma}(\otimes)$R, $\rightsquigarrow$ L We deal with this exactly like the case $\otimes$R, $\rightsquigarrow$ L (Page 22) of Theorem 3.

$\boldsymbol{\sigma}(\oplus)$R, $\rightsquigarrow$ L Here the proof looks like

$$\dfrac{\dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma \vdash \boldsymbol{\sigma}(A),\, \Delta\end{matrix}}{\Gamma \vdash \boldsymbol{\sigma}(A \oplus B),\, \Delta} \qquad \dfrac{\begin{matrix}\Pi'\\ \vdots\\ \Gamma',\, \boldsymbol{\sigma}(A_1) \vdash \Diamond C,\, \Delta'\end{matrix}}{\Gamma',\, \boldsymbol{\sigma}(A \oplus B) \vdash \Diamond C,\, \Delta'}}{\Gamma,\, \Gamma' \vdash \Diamond C,\, \Delta,\, \Delta'}\,\text{cut}$$

Now we have $\boldsymbol{\sigma}(A) \rightsquigarrow \boldsymbol{\sigma}(A \oplus B)$; we also have $\boldsymbol{\sigma}(A \oplus B) \rightsquigarrow \boldsymbol{\sigma}(A_1)$; so we have $\boldsymbol{\sigma}(A) \rightsquigarrow \boldsymbol{\sigma}(A_1)$ and we can rewrite our proof as follows:

$$\dfrac{\begin{matrix}\Pi\\ \vdots\\ \Gamma \vdash \boldsymbol{\sigma}(A),\, \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\Pi'\\ \vdots\\ \Gamma,\, \boldsymbol{\sigma}(A_1) \vdash \Diamond C,\, \Delta\end{matrix}}{\Gamma',\, \boldsymbol{\sigma}(A) \vdash \Diamond C,\, \Delta}}{\Gamma,\, \Gamma \vdash \Diamond C,\, \Delta}\,\text{cut}$$

Cuts with $\boldsymbol{\tau}(A)$ We now deal with the cases where the cutformula is of the form $\boldsymbol{\tau}(A)$ and is principal on both sequents.

$\boldsymbol{\tau}(\rightsquigarrow)$R, $\boldsymbol{\tau}(\rightsquigarrow)$L We have

$$\dfrac{\dfrac{\left\{\begin{matrix}\Pi_i\\ \vdots\\ \Gamma \vdash \boldsymbol{\sigma}(A_i),\, \Delta\end{matrix}\right\}_{\boldsymbol{\sigma}(A) \rightsquigarrow \boldsymbol{\sigma}(A_i)}}{\Gamma \vdash \boldsymbol{\tau}(A),\, \Delta} \qquad \dfrac{\begin{matrix}\Pi'\\ \vdots\\ \Gamma',\, \boldsymbol{\sigma}(A_i) \vdash \Delta'\end{matrix}}{\Gamma',\, \boldsymbol{\tau}(A) \vdash \Delta'}}{\Gamma,\, \Gamma' \vdash \Delta,\, \Delta'}\,\text{cut}$$

which we replace with

$$\dfrac{\begin{matrix}\Pi_i\\ \vdots\\ \Gamma \vdash \boldsymbol{\sigma}(A_i),\, \Delta\end{matrix} \qquad \begin{matrix}\Pi'\\ \vdots\\ \Gamma',\, \boldsymbol{\sigma}(A_i),\, \Delta'\end{matrix}}{\Gamma,\, \Gamma' \vdash \Delta,\, \Delta'}\,\text{cut}$$

$\boldsymbol{\tau}(\rightsquigarrow)$R, $\boldsymbol{\tau}(\otimes)$L Here our proof looks like

$$\dfrac{\dfrac{\left\{\begin{array}{c}\Pi_i\\ \vdots\\ \Gamma \vdash \boldsymbol{\sigma}(C_i),\ \Delta\end{array}\right\}_{\boldsymbol{\sigma}(A\otimes B)\rightsquigarrow\boldsymbol{\sigma}(C_i)}}{\Gamma \vdash \boldsymbol{\tau}(A\otimes B),\ \Delta} \qquad \dfrac{\begin{array}{c}\Pi'\\ \vdots\\ \Gamma',\ \boldsymbol{\tau}(A)\otimes\boldsymbol{\tau}(B) \vdash \Delta'\end{array}}{\Gamma',\ \boldsymbol{\tau}(A\otimes B)\vdash \Delta'}}{\Gamma,\ \Gamma' \vdash \Delta,\ \Delta'}\ \text{cut}$$

Consider the sequent $\Gamma \vdash \boldsymbol{\tau}(A)\otimes\boldsymbol{\tau}(B),\ \Delta$; if we first apply the rule $\boldsymbol{\tau}(\rightsquigarrow)$ $\otimes$ R to it, we have premises $\Gamma \vdash \boldsymbol{\sigma}(A_j)\otimes\boldsymbol{\tau}(B),\ \Delta$ where the $\boldsymbol{\sigma}(A_j)$ are all the rewrites of $\boldsymbol{\sigma}(A)$. If we then apply the same rule, but to $\boldsymbol{\tau}(B)$ in those premises, we find we have premises $\Gamma \vdash \boldsymbol{\sigma}(A_i)\otimes\boldsymbol{\sigma}(B_j)$. However, note that each $\boldsymbol{\sigma}(A_i)\otimes\boldsymbol{\sigma}(B_j)$ is equivalent to a $\boldsymbol{\sigma}(A_i\otimes B_j)$, which is a rewrite of $\boldsymbol{\sigma}(A\otimes B)$; so, among the proofs Π_i, there must be a suitable proof $\Pi_{j,k}$ of $\Gamma \vdash \boldsymbol{\sigma}(A_j\otimes B_k),\ \Delta$. So, finally, we have the proof in Table 8. Here the bottom cut is higher up on the right, but lower on the left; a suitable definition of cut rank will allow us to make an induction.

$\boldsymbol{\tau}(\rightsquigarrow)$R, $\boldsymbol{\tau}(\oplus)$L We have

$$\dfrac{\dfrac{\left\{\begin{array}{c}\Pi_{i,j}\\ \vdots\\ \Gamma \vdash \boldsymbol{\sigma}(A_i)\oplus\boldsymbol{\sigma}(B_j,\ \Delta)\end{array}\right\}_{\boldsymbol{\sigma}(A)\rightsquigarrow\boldsymbol{\sigma}(A_i),\boldsymbol{\sigma}(B)\rightsquigarrow\boldsymbol{\sigma}(B_j)}}{\Gamma \vdash \boldsymbol{\tau}(A\oplus B),\ \Delta} \qquad \begin{array}{c}\Pi''\\ \vdots\\ \Gamma',\boldsymbol{\tau}(A\oplus B)\vdash \Delta'\end{array}}{\Gamma,\ \Gamma' \vdash \Delta,\ \Delta'}\ \text{cut}$$

We apply Lemma 4 to Π' and find a proof Π'' of $\Gamma', \boldsymbol{\sigma}(A_i\oplus B_j$ for some i and j. So we can rewrite the proof

$$\dfrac{\begin{array}{c}\Pi_{i,j}\\ \vdots\\ \Gamma \vdash \boldsymbol{\sigma}(A_i\oplus B_j),\ \Delta\end{array} \qquad \begin{array}{c}\Pi''\\ \vdots\\ \Gamma',\ \boldsymbol{\sigma}(A_i\oplus B_j)\vdash \Delta'\end{array}}{\Gamma,\ \Gamma' \vdash \Delta,\ \Delta'}\ \text{cut}$$

$\boldsymbol{\tau}(\rightsquigarrow)\otimes$ R, $\otimes$L We have

$$\dfrac{\dfrac{\left\{\begin{array}{c}\Pi_i\\ \vdots\\ \Gamma \vdash \boldsymbol{\sigma}(A_i)\otimes B,\ \Delta\end{array}\right\}_{\boldsymbol{\sigma}(A)\rightsquigarrow\boldsymbol{\sigma}(A_i)}}{\Gamma \vdash \boldsymbol{\tau}(A)\otimes B,\ \Delta} \qquad \dfrac{\begin{array}{c}\Pi'\\ \vdots\\ \Gamma',\ \boldsymbol{\tau}(A),\ B \vdash \Delta'\end{array}}{\Gamma',\ \boldsymbol{\tau}(A)\otimes B\vdash \Delta'}}{\Gamma,\ \Gamma' \vdash \Delta,\ \Delta'}\ \text{cut}$$

We can now apply Lemma 4 to Π' and find a proof Π'' of $\Gamma',\ \boldsymbol{\sigma}(A_i),\ B \vdash \Delta'$, for some suitable A_i. So we can now rewrite our proof as

$$\dfrac{\begin{array}{c}\Pi_i\\ \vdots\\ \Gamma \vdash \boldsymbol{\sigma}(A_i)\otimes B,\ \Delta\end{array} \qquad \dfrac{\begin{array}{c}\Pi''\\ \vdots\\ \Gamma,\boldsymbol{\sigma}(A_i),\ B \vdash \Delta'\end{array}}{\Gamma',\boldsymbol{\sigma}(A_i)\otimes B\vdash \Delta'}}{\Gamma,\ \Gamma' \vdash \Delta,\ \Delta'}\ \text{cut}$$

Table 8 The Case $\boldsymbol{\tau}(\cdot)$R, $\boldsymbol{\tau}(\otimes)$L

$$
\dfrac{
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\left\{
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\left\{
\dfrac{
\begin{array}{c}\Pi_{j,k}\\ \vdots \\ \Gamma \vdash \boldsymbol{\sigma}(A_j \otimes B_k),\, \Delta\end{array}
\qquad
\dfrac{\dfrac{}{\boldsymbol{\sigma}(A_j) \otimes \boldsymbol{\sigma}(B_k) \vdash \boldsymbol{\sigma}(A_j) \otimes \boldsymbol{\sigma}(B_k)}}{\boldsymbol{\sigma}(A_j \otimes B_k) \vdash \boldsymbol{\sigma}(A_j) \otimes \boldsymbol{\sigma}(B_k)}
}{\Gamma \vdash \boldsymbol{\sigma}(A_j) \otimes \boldsymbol{\sigma}(B_k),\, \Delta}\,\text{cut}
\right\}_{\boldsymbol{\sigma}(B) \rightsquigarrow \boldsymbol{\sigma}(B_k)}
}{\Gamma \vdash \boldsymbol{\sigma}(A_j) \otimes \boldsymbol{\tau}(B), \Delta}
\right\}_{\boldsymbol{\sigma}(A) \rightsquigarrow \boldsymbol{\sigma}(A_j)}
}{\Gamma \vdash \boldsymbol{\tau}(A) \otimes \boldsymbol{\tau}(B),\, \Delta}
\qquad
\begin{array}{c}\Pi'\\ \vdots \\ \Gamma',\, \boldsymbol{\tau}(A) \otimes \boldsymbol{\tau}(B) \vdash \Delta'\end{array}
}{\Gamma,\, \Gamma' \vdash \Delta,\, \Delta'}\,\text{cut}
$$

which moves the cut upwards on the left.

2.4 Applications

Here we compile a list of results about inference in the various systems. Some of them are elementary, whereas others (the "corollaries") use cut elimination.

Corollary The only valid instances of $A_1, \ldots, A_r \vdash \Diamond B$, in either $\mathsf{LL}^{\rightsquigarrow}_{\Diamond}0$ or $\mathsf{LL}^{\rightsquigarrow}_{\Diamond}1$, and in which A_i and B are state propositions, are those in which $(A_1 \otimes \ldots \otimes A_r) \rightsquigarrow B$.

Proof We must apply the $\Diamond$R rule at some stage, at which time we must prove an entailment of the form

$$A',\, A'',\, \ldots \vdash B', \tag{7}$$

where the As and B' are state propositions, $B' \rightsquigarrow B$, and

$$A_1 \otimes \ldots \otimes A_r \begin{cases} \cong A' \otimes A'' \ldots & \text{in } \mathsf{LL}^{\rightsquigarrow}_{\Diamond}0 \\ \rightsquigarrow A' \otimes A'' \ldots & \text{in } \mathsf{LL}^{\rightsquigarrow}_{\Diamond}1 \end{cases}$$

Now (7) is only valid if $A' \otimes A' \otimes \ldots \cong B''$, so we have the result.

Proposition In all three systems,

$$\begin{aligned} A, \Diamond B &\ \vdash\ \Diamond(A \otimes B) \\ \Diamond A, \Diamond B &\ \vdash\ \Diamond(A \otimes B) \end{aligned}$$

Proof Obvious.

Corollary In $\mathsf{LL}^{\rightsquigarrow}_{\Diamond}0$, the only valid sequents of the form

$$A,\, A' \multimap B' \vdash \Diamond B,$$

where A, A', B and B' are state propositions, are those in which there is an A'' with $A \cong A' \otimes A''$ and $(A'' \otimes B') \rightsquigarrow B$.

Proof Consider a cut free proof. We must use $\Diamond$R before we use $\otimes$R, and we might well have to use $\otimes$L before any of the environment splitting rules; but otherwise the order of application of the rules is pretty well immaterial. So a representative proof will be

$$\dfrac{\dfrac{\dfrac{}{A' \vdash A'} \qquad \dfrac{\dfrac{\dfrac{}{A'' \vdash A''} \qquad \dfrac{}{B' \vdash B'}}{A'',\, B' \vdash A'' \otimes B'}}{A'',\, B' \vdash \Diamond B}\,\Diamond\text{R}}{A',\, A'',\, A' \multimap B' \vdash \Diamond B}\,\multimap\text{L}}{A,\, A' \multimap B \vdash \Diamond B}\,\otimes\text{L}$$

and the constraints on the state propositions are evident.

Corollary In $\mathsf{LL}_{\Diamond}^{\leadsto}1$, the only valid sequents of the form

$$A,\, A' \multimap B' \vdash \Diamond B,$$

where A, A', B and B' are state propositions, are those in which there is an A'' with $A \leadsto (A' \otimes A'')$ and $(A'' \otimes B') \leadsto B$.

Proof Much the same, except that we are now allowed to use $\leadsto$ L, so proofs will now look like:

$$\dfrac{\dfrac{\dfrac{\overline{A' \vdash A'} \quad \dfrac{\dfrac{\overline{A'' \vdash A''} \quad \overline{B' \vdash B'}}{A'',\, B' \vdash A'' \otimes B'}}{A'',\, B' \vdash \Diamond B}\,\Diamond\text{R}}{A',\, A'',\, A' \multimap B' \vdash \Diamond B}\,\multimap \text{L}}{A' \otimes A'',\, A' \multimap B \vdash \Diamond B}\,\otimes\text{L}}{A,\, A' \multimap B \vdash \Diamond B}\,\leadsto \text{L}$$

(applications of $\leadsto$ L later in the proof can be incorporated either into the first one or into the application of $\Diamond$R).

Remark 5 Corollary 2 almost gives a proof-theoretic treatment of Thielscher's algorithm in [24, 25]. He there describes a treatment of ramification in which one first computes the direct effects of actions using the "fluent calculus" [23] (by [7] this is equivalent to non-modal linear logic), and then applies a rewrite system to compute the ramification. We can obtain the same effect by applying Corollary 2 to inferences in which A is an input situation, B an output situation, and $A' \multimap B'$ is an action. The proof of the corollary shows that our proof search can be split into two phases: a non-modal phase, which computes the direct effects of the action, and a modal phase, which computes the ramification. This *almost* translates Thielscher's algorithm into proof theory; what we cannot yet handle is detecting whether the sequence of ramifications has terminated. Our proofs, that is, sequents describe sequences of ramifications of any length, and not just the sequences which terminate.

Remark 6 Corollary 6 generalises this algorithm; it shows that it is possible to apply rewrites both to the input situation and to the output situation. This is a much less synchronised mixture of action and ramification than that described in Corollary 2. As we shall see, one of the main differences between $\mathsf{LL}_{\Diamond}^{\leadsto}0$ and $\mathsf{LL}_{\Diamond}^{\leadsto}1$ is that the latter is less synchronised than the former.

Corollary In $\mathsf{LL}_{\Diamond,\boldsymbol{\sigma},\boldsymbol{\tau}}^{\leadsto}$, the only valid proofs of $\boldsymbol{\sigma}(A), \boldsymbol{\sigma}(A') \multimap \boldsymbol{\sigma}(B') \vdash \Diamond\boldsymbol{\tau}(B)$, where B is a tensor product of atoms, are those in which there is an A'' with $\boldsymbol{\sigma}(A) \leadsto \boldsymbol{\sigma}(A' \otimes A'')$ and $\boldsymbol{\sigma}(A'' \otimes B') \leadsto \boldsymbol{\sigma}(B)$ and in which B is terminal (i.e. there are no non-trivial rewrites from B).

Proof Proofs of the sequent in question are now going to be of the form

$$
\dfrac{\dfrac{}{\boldsymbol{\sigma}(A') \vdash A'} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\boldsymbol{\sigma}(B) \vdash \boldsymbol{\sigma}(B)}}{\boldsymbol{\sigma}(B) \vdash \boldsymbol{\tau}(B)} \leadsto (\boldsymbol{\tau})\mathrm{R}}{\boldsymbol{\sigma}(B) \vdash \Diamond\boldsymbol{\tau}(B)} \Diamond\mathrm{R}}{\boldsymbol{\sigma}(A''),\, \boldsymbol{\sigma}(B') \vdash \Diamond\boldsymbol{\tau}(B)} \leadsto \mathrm{L}}{\dfrac{\boldsymbol{\sigma}(A'),\, \boldsymbol{\sigma}(A''),\, \boldsymbol{\sigma}(A') \multimap \boldsymbol{\sigma}(B') \vdash \Diamond\boldsymbol{\tau}(B)}{\dfrac{\boldsymbol{\sigma}(A' \otimes A''),\, \boldsymbol{\sigma}(A') \multimap \boldsymbol{\sigma}(B) \vdash \Diamond\boldsymbol{\tau}(B)}{\boldsymbol{\sigma}(A),\, \boldsymbol{\sigma}(A') \multimap \boldsymbol{\sigma}(B) \vdash \Diamond\boldsymbol{\tau}(B)} \leadsto \mathrm{L}} \otimes\mathrm{L}} \multimap \mathrm{L}
$$

Example 1 (Balls and String) We quote the scenario from [30]:

> There are two balls (b_1 and b_2) linked together with a string s of length 1. There are three locations: l_0 is on the surface of a table, l_1 is one unit above it, and l_2 is two units above l_0. At time 0 the two balls are at l_0 and the string is slack; b_2 is then raised so that, at time1, it is at l_2 and, in consequence, b_1 is hanging below it at l_1.

We can handle this with the rewrites

$$
\begin{array}{rcl}
\mathrm{at}(b, l_i) \otimes \mathrm{at}(b', l_j) & \leadsto & \mathrm{at}(b, l_i) \otimes \mathrm{at}(b', l_{i-1}) \\
 & & i = 1, 2, 3, j < i \\
\mathrm{at}(b, l_i) \otimes \mathrm{at}(b', l_j) \otimes \mathrm{slack}(s) & \leadsto & \mathrm{at}(b, l_i) \otimes \mathrm{at}(b', l_j) \otimes \mathrm{taut}(s) \\
 & & i, j = 0, 1, 2, 3, i \neq j \\
\mathrm{at}(b, l_i) \otimes \mathrm{at}(b', l_i) \otimes \mathrm{taut}(s) & \leadsto & \mathrm{at}(b, l_i) \otimes \mathrm{at}(b', l_i) \otimes \mathrm{slack}(s) \\
 & & i = 0, 1, 2, 3
\end{array}
$$

and we look for proofs of the sequent

$$
\boldsymbol{\sigma}(\mathrm{at}(b_1, l_0) \otimes \mathrm{at}(b_2, l_0) \otimes \mathrm{slack}(s)),\, \boldsymbol{\sigma}(\mathrm{at}(b_2, l_0)) \multimap \boldsymbol{\sigma}(\mathrm{at}(b_2, l_2)) \vdash \Diamond\boldsymbol{\tau}(X),
$$

where X is a tensor product of atoms; because of Corollary 4, the only such proofs are those in which X corresponds to the correct final situation.

3 Semantics

3.1 Phase Semantics

We now describe a semantics for our modal operators; this will be modelled on Girard's phase semantics [5, 4].

Definition 7 A *phase frame* is a tuple $\mathfrak{M} = \langle M, \circ, \mathbb{I}, \bot, (\cdot)^\circ \rangle$, where $\langle M, \circ, \mathbb{I} \rangle$ is a commutative monoid with unit, and where, if we define

$$
\begin{array}{rcl}
m \circ X & \stackrel{\mathrm{def}}{=} & \{m \circ x \mid x \in X\} \\
m \sqsubseteq m' & \stackrel{\mathrm{def}}{=} & \forall n \in M.\, m \circ n \in \bot \to m' \circ n \in \bot,
\end{array}
$$

(for $X \subseteq M$ and $m, m' \in M$) we have

Table 9 Phase Semantics

$$
\begin{array}{rcl}
[\![A \multimap B]\!] & = & [\![A]\!] \multimap [\![B]\!] \\
[\![A^\perp]\!] & = & [\![A]\!]^\perp \\
[\![A \otimes B]\!] & = & ([\![A]\!] \circ [\![B]\!])^{\perp\perp} \\
[\![A \parr B]\!] & = & ([\![A]\!]^\perp \circ [\![B]\!]^\perp)^\perp \\
[\![\perp]\!] & = & \perp \\
[\![\mathbf{1}]\!] & = & \{\mathbb{I}\}^{\perp\perp} \\
[\![A \oplus B]\!] & = & ([\![A]\!] \cup [\![B]\!])^{\perp\perp} \\
[\![A \& B]\!] & = & [\![A]\!] \cap [\![B]\!] \\
[\![\mathbf{0}]\!] & = & \emptyset^{\perp\perp} \\
[\![\top]\!] & = & M \\
[\![\Box A]\!] & = & ([\![A]\!]^\circ)^{\perp\perp} \\
[\![\Diamond A]\!] & = & ([\![A^\perp]\!]^\circ)^\perp
\end{array}
$$

- $\perp$ is a subset of M downward closed under $\sqsubseteq$, and
- $(\cdot)^\circ$ is a map $\mathcal{P}(M) \to \mathcal{P}(M)$ such that

$$
\begin{array}{rcll}
\perp^\circ & \subseteq & \perp & (8) \\
A \subseteq B & \Rightarrow & A^\circ \subseteq B^\circ & (9) \\
m \circ (A^\circ) & \subseteq & (m \circ A)^\circ & (10) \\
m \in A^\circ & \Rightarrow & \exists m' \in (A^\circ)^\circ.\, m \sqsubseteq m' & (11)
\end{array}
$$

  for all $m \in M$, $A \in \mathcal{P}M$.

A good way of getting such an operation $(\cdot)^\circ$ is the following:

Definition 8 A *relational phase frame* is a tuple $\mathfrak{M} = \langle M, \circ, \mathbb{I}, \perp, \boldsymbol{\rho}\rangle$, where $M, \circ, \mathbb{I}, \perp$ are as above and $\boldsymbol{\rho}$ is a relation on M such that

- $x\,\boldsymbol{\rho}\,y \wedge x \in \perp \;\to\; y \in \perp$,
- $x\,\boldsymbol{\rho}\,y \;\to\; (x \circ m)\,\boldsymbol{\rho}\,(y \circ m)$, and
- $x\,\boldsymbol{\rho}\,y \;\to\; \exists z, z'.\, x\,\boldsymbol{\rho}\,z \wedge z\,\boldsymbol{\rho}\,z' \wedge y \sqsubseteq z'$.

Given a relational phase frame, we define the *associated phase frame* by adjoining to $M, \circ, \mathbb{I}, \perp$ the operation $A \mapsto A^\circ = \{x \mid \forall y.\, x\,\boldsymbol{\rho}\,y \;\to\; y \in A\}$.

Remark 7 It is elementary to prove that the associated frame of a relational phase frame is, in fact, a phase frame.

We now define the following operations on $\mathcal{P}M$:

Definition 9 If $X, Y \subseteq M$, we define

$$
\begin{array}{rcl}
X \multimap Y & = & \{m \in M \mid \forall x \in X.\, m \circ x \in Y\} \\
X^\perp & = & X \multimap \perp
\end{array}
$$

Having defined frames, we can now give semantic values to formulae; this definition is closely analogous to [5, p. 23].

Definition 10 Subsets of the form $X^\perp$ are called *facts*. A semantic valuation is given by an assignment $A \mapsto [\![A]\!]$ of atomic propositions to facts of M, and is extended to the whole of the language by the clauses in Table 9.

Lemma For $m, m', n \in M$,

$$m \sqsubseteq m' \;\rightarrow\; m \circ n \sqsubseteq m' \circ n$$

Proof Obvious.

Lemma For $A, B \in \mathcal{P}M$ with B downward closed, $A \multimap B$ is downward closed.

Proof Suppose $m \in A \multimap B$, and suppose $m' \sqsubseteq m$. Then, for any $a \in A$, $m' \circ a \sqsubseteq m \circ a \in B$, and B is downward closed.

Lemma Facts are downward closed.

Proof $\perp$ is downward closed.

Lemma For any formula A of the language, $[\![A]\!]$ is a fact.

Proof A straightforward induction.

Lemma For A a fact, $A^\circ \subseteq A$.

Proof We first establish that $A^\perp \subseteq A^{\circ\perp}$. Suppose $m \in A^\perp$, and suppose $n \in A^\circ$. Then, since $m \circ A \subseteq \perp$, $m \circ n \in m \circ A^\circ \subseteq m \circ A^\circ \subseteq \perp^\circ \subseteq \perp$, so $m \in A^{\circ\perp}$.

So now we argue as follows: by the contravariance of $(\cdot)^\perp$, we have $A^{\circ\perp\perp} \subseteq A^{\perp\perp}$, but $A^\circ \subseteq A^{\circ\perp\perp}$, which gives the result.

Lemma For A and B propositions, $[\![A \parr B]\!]^\perp = [\![A^\perp \otimes B^\perp]\!]$

Proof Immediate from the definitions of $[\![\cdot \otimes \cdot]\!]$ and $[\![\cdot \parr \cdot]\!]$.

Lemma For $X \subseteq M$, $X^\perp = X^{\perp\perp\perp}$.

Proof Clearly, for any $Y \subseteq M$, $Y \subseteq Y^{\perp\perp}$, so we have $X^\perp \subseteq X^{\perp\perp\perp}$. However, by definition, $X^{\perp\perp} \circ (X^{\perp\perp})^\perp \subseteq \perp$, and $X \subseteq X^{\perp\perp}$, so we have $X \circ (X^{\perp\perp})^\perp \subseteq \perp$, and so $X^{\perp\perp\perp} \subseteq X^\perp$.

Lemma For $X \subseteq M$, $X^{\perp\perp}$ is the smallest fact containing X.

Proof (See [4, Lemma 1.13.1].) Clearly $X^{\perp\perp}$ is a fact containing X. Suppose Y is a fact containing X. Since Y is a fact, $Y = Z^{\perp\perp}$ for some Z; but then $Y^{\perp\perp} = Y^{\perp\perp\perp} = Y^\perp = Y$. So we have

$$\begin{array}{rrcll} & X & \subseteq & Y & \\ \Rightarrow & Y^\perp & \subseteq & X^\perp & \\ \Rightarrow & X^{\perp\perp} & \subseteq & Y^{\perp\perp} & = \;Y \end{array}$$

Lemma For $X, Y, Z \subseteq M$, $(X \multimap Z) \cap (Y \multimap Z) = (X \cup Y) \multimap Z$.

Proof Immediate from the definition.

Lemma If $X, Y \subseteq M$ are facts, then $X \cap Y$ is a fact.

Proof Use Lemma 13.

Lemma For propositions A and B, $[\![A \oplus B]\!]^\perp = [\![A^\perp \& B^\perp]\!]$ and $[\![A \& B]\!]^\perp = [\![A^\perp \oplus B^\perp]\!]$.

Proof For the first,

$$\begin{aligned}[\![A \oplus B]\!]^\perp &= [\![A]\!] \cup [\![B]\!]^{\perp\perp\perp} \\ &= [\![A]\!] \cup [\![B]\!]^\perp \\ &= [\![A]\!]^\perp \cap [\![B]\!]^\perp \\ &= [\![A^\perp \& B^\perp]\!]\,;\end{aligned}$$

this yields the second on dualising.

Lemma For any formula A, $[\![A]\!] = [\![A^{\perp\perp}]\!]$.

Proof By Lemma 8, $[\![A]\!]$ is a fact; by Lemma 11, $[\![A]\!] = [\![A]\!]^{\perp\perp}$; and, by the definition of $[\![(\cdot)^\perp]\!]$, $[\![A]\!]^{\perp\perp} = [\![A^{\perp\perp}]\!]$.

Lemma For A, B and C formulae, we have

$$[\![A]\!] \subseteq [\![B \parr C]\!] \quad \text{iff} \quad [\![A \otimes B^\perp]\!] \subseteq [\![C]\!]\,.$$

Proof Since $[\![A \otimes B^\perp]\!]$ is the smallest fact containing $[\![A]\!] \circ [\![B^\perp]\!]$, it suffices to prove that

$$[\![A]\!] \subseteq [\![B \parr C]\!] \quad \text{iff} \quad [\![A]\!] \circ [\![B^\perp]\!] \subseteq [\![C]\!]\,.$$

Firstly, suppose that $[\![A]\!] \subseteq [\![B \parr C]\!]$. For any $a \in A$, $a \in [\![B \parr C]\!] = ([\![B^\perp]\!] \circ [\![C^\perp]\!])^\perp$ iff, for all $m \in [\![B^\perp]\!]$ and $n \in [\![C^\perp]\!]$, $a \circ m \circ n \in \perp$. Now let $a \circ m \in [\![A]\!] \circ [\![B^\perp]\!]$; we show that $a \circ m \in [\![C]\!]^{\perp\perp}$, which will imply (since $[\![C]\!]$ is a fact) that $a \circ m \in [\![C]\!]$. So let $n \in [\![C]\!]^\perp$; we know that $a \circ m \circ n \in \perp$, which is what we wanted to show.

Conversely, suppose that $[\![A]\!] \circ [\![B^\perp]\!] \subseteq [\![C]\!]$, and suppose that $a \in [\![A]\!]$; we must show that, for any $m \in [\![B^\perp]\!]$ and $n \in [\![C^\perp]\!]$, $a \circ m \circ n \in \perp$. But $a \circ m \in [\![C]\!] = [\![C]\!]^{\perp\perp}$, so the result is immediate.

Lemma For A, B and C formulae, we have

$$[\![A]\!] \subseteq [\![B^\perp \parr C]\!] \quad \text{iff} \quad [\![A \otimes B]\!] \subseteq [\![C]\!]\,.$$

Proof Use Lemma 17 together with Lemma 16.

Lemma For $A \subseteq M$, $A^{\circ\perp\perp} = A^{\circ\circ\perp\perp}$.

Proof Clearly $A^{\circ\circ\perp\perp} \subseteq A^{\circ\perp\perp}$. For the converse, it suffices to prove that $A^{\circ\perp} \subseteq A^{\circ\circ\perp}$. So suppose that $m \in A^{\circ\perp}$, and consider an arbitrary $a \in A^{\circ\circ}$; we want to prove that $m \circ a \in \perp$. But, by (11), there is an $a' \in A^\circ$ with $a' \sqsubseteq a$, and certainly $a' \circ m \in \perp$; so $a \circ m \in \perp$.

Now we can prove soundness and completeness.

Table 10 The System $\mathsf{LL}_\Diamond{}''$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \Box A \vdash \Delta}\,\Box\mathrm{L} \qquad \frac{\Gamma, \Box A \vdash B,\ \Delta}{\Gamma, \Box A \vdash \Box B,\ \Delta}\,\Box\mathrm{R}$$

$$\frac{}{\Diamond A \vdash (\Box A^\perp)^\perp}\,\Diamond\mathrm{L} \qquad \frac{}{(\Box A^\perp)^\perp \vdash \Diamond A}\,\Diamond\mathrm{R}$$

3.1.1 Soundness

Theorem (Soundness for the Phase Semantics) If $\Gamma \vdash \Delta$ is a valid sequent, then $\left[\!\left[\bigotimes_{\gamma\in\Gamma} \gamma\right]\!\right] \subseteq \left[\!\left[\bigparr_{\delta\in\Delta} \gamma\right]\!\right]$.

Proof We make an induction on the length of the proof of $\vdash \Delta$; we go by cases according to the last rule of the proof. We use a presentation of our basic system, $\mathsf{LL}_\Diamond{}''$, given in Table 10; we can easily prove a dual form of Proposition 9 to show that they are equivalent to the rules of Table varModalRules.

Axiom The sequent is of the form $A \ \vdash \ A$, so the result is clear.

$\perp$ **R** Note that $[\![\perp \parr A]\!] = ([\![\perp]\!]^\perp \circ [\![A]\!]^\perp)^\perp$, $[\![\perp]\!] = \perp$, and $\perp^\perp = \mathbb{I}^{\perp\perp}$; so we have $[\![\perp \parr A]\!] = [\![A]\!]$. Consequently, by the compositionality of $[\![\cdot]\!]$, if we have

$$\left[\!\left[\bigotimes_{\gamma\in\Gamma} \gamma\right]\!\right] \subseteq \left[\!\left[\bigparr_{\delta\in\Delta} \delta\right]\!\right]$$

then we have

$$\left[\!\left[\bigotimes_{\gamma\in\Gamma} \gamma\right]\!\right] \subseteq \left[\!\left[\perp \parr (\bigparr_{\delta\in\Delta} \delta)\right]\!\right]$$

since the two right hand sides are equal.

$\perp$ **L** The sequent must be $\perp\vdash$, and we have to prove $[\![\perp]\!] \subseteq [\![\perp]\!]$.

1R The sequent must be $\vdash \ \mathbf{1}$, and we have to prove $[\![\mathbf{1}]\!] \subseteq [\![\mathbf{1}]\!]$.

1L We argue, as for $\perp$ R, that $[\![A]\!] = [\![A \otimes \mathbf{1}]\!]$, and then the result is clear.

$\parr$**R** Immediate, since $\bigparr_{\delta\in\Delta} \delta$ is unchanged.

$\otimes$**L** Likewise immediate.

$\otimes$**R** We can assume that

$$\left[\!\left[\bigotimes_{\gamma\in\Gamma} \gamma\right]\!\right] \subseteq \left[\!\left[A \parr (\bigparr_{\delta\in\Delta} \delta\right]\!\right]$$

and

$$\left[\!\left[\bigotimes_{\gamma\in\Gamma'} \gamma\right]\!\right] \subseteq \left[\!\left[B \parr (\bigparr_{\delta\in\Delta'} \delta\right]\!\right]$$

and we have to prove

$$\left[\!\!\left[\bigotimes_{\gamma\in\Gamma\cup\Gamma'}\gamma\right]\!\!\right] \subseteq \left[\!\!\left[(A\otimes B)\parr(\bigparr_{\delta\in\Delta\cup\Delta'}\delta)\right]\!\!\right].$$

By a repeated application of Lemma 17, we can assume that we are given

$$\left[\!\!\left[(\bigotimes_{\gamma\in\Gamma}\gamma)\otimes(\bigotimes_{\delta\in\Delta}\delta^{\perp})\right]\!\!\right] \subseteq [\![A]\!]$$

and

$$\left[\!\!\left[(\bigotimes_{\gamma\in\Gamma'}\gamma)\otimes(\bigotimes_{\delta\in\Delta'}\delta^{\perp})\right]\!\!\right] \subseteq [\![B]\!]$$

and that we have to prove

$$\left[\!\!\left[(\bigotimes_{\gamma\in\Gamma\cup\Gamma'}\gamma)\otimes(\bigotimes_{\delta\in\Delta\cup\Delta'}\delta^{\perp})\right]\!\!\right] \subseteq [\![A\otimes B]\!].$$

It suffices to prove that

$$\left[\!\!\left[(\bigotimes_{\gamma\in\Gamma}\gamma)\otimes(\bigotimes_{\delta\in\Delta}\delta^{\perp})\right]\!\!\right] \circ \left[\!\!\left[(\bigotimes_{\gamma\in\Gamma'}\gamma)\otimes(\bigotimes_{\delta\in\Delta'}\delta^{\perp})\right]\!\!\right] \subseteq [\![A\otimes B]\!];$$

by the definition of $[\![\cdot\otimes\cdot]\!]$, this is immediate.

$\parr$L We can assume that we are given

$$\left[\!\!\left[(\bigotimes_{\gamma\in\Gamma}\gamma)\otimes A\right]\!\!\right] \subseteq \left[\!\!\left[\bigparr_{\delta\in\Delta}\delta\right]\!\!\right]$$

and

$$\left[\!\!\left[(\bigotimes_{\gamma\in\Gamma'}\gamma)\otimes B\right]\!\!\right] \subseteq \left[\!\!\left[\bigparr_{\delta\in\Delta'}\delta\right]\!\!\right]$$

and we have to prove

$$\left[\!\!\left[(\bigotimes_{\gamma\in\Gamma\cup\Gamma'}\gamma)\otimes(A\parr B)\right]\!\!\right] \subseteq \left[\!\!\left[\bigparr_{\delta\in\Delta\cup\Delta'}\delta\right]\!\!\right].$$

By Lemma 18 and Lemma 10, we can reduce this to $\otimes R$.

&R By similar reductions to the above, we have to show that, if

$$\left[\!\!\left[\bigotimes_{\gamma\in\Gamma}\gamma\right]\!\!\right] \subseteq [\![A]\!]$$

and

$$\left[\!\!\left[\bigotimes_{\gamma\in\Gamma}\gamma\right]\!\!\right] \subseteq [\![B]\!]$$

then

$$\left[\!\left[\bigotimes_{\gamma\in\Gamma}\gamma\right]\!\right] \subseteq [\![A\& B]\!]\,;$$

but this is immediate from the definition of $[\![\cdot\&\cdot]\!]$.

$\oplus$L We use Lemma 15 to reduce this to the previous case.

$\oplus$R If we have

$$\left[\!\left[\bigotimes_{\gamma\in\Gamma}\gamma\right]\!\right] \subseteq [\![A]\!]$$

then

$$\left[\!\left[\bigotimes_{\gamma\in\Gamma}\gamma\right]\!\right] \subseteq [\![A\oplus B]\!]$$

because, by definition of $[\![\cdot\oplus\cdot]\!]$, $[\![A]\!]\subseteq[\![A\oplus B]\!]$.

&L Again we reduce this to the previous case.

$\Box$L We have to prove that, if

$$\left[\!\left[(\bigotimes_{\gamma\in\Gamma}\gamma)\otimes A\right]\!\right] \subseteq \left[\!\left[\bigotimes_{\delta\in\Delta}\delta\right]\!\right]$$

then

$$\left[\!\left[(\bigotimes_{\gamma\in\Gamma}\gamma)\otimes \Box A\right]\!\right] \subseteq \left[\!\left[\bigotimes_{\delta\in\Delta}\delta\right]\!\right],$$

or, equivalently, that if

$$\left[\!\left[\bigotimes_{\gamma\in\Gamma}\gamma\right]\!\right]\circ[\![A]\!] \subseteq \left[\!\left[\bigotimes_{\delta\in\Delta}\delta\right]\!\right]$$

then

$$\left[\!\left[\bigotimes_{\gamma\in\Gamma}\gamma\right]\!\right]\circ[\![\Box A]\!] \subseteq \left[\!\left[\bigotimes_{\delta\in\Delta}\delta\right]\!\right];$$

but the latter equivalence is trivially true since, by (8), $[\![\Box A]\!]\subseteq[\![A]\!]$.

$\Box$R By the usual arguments, we can reduce this to showing that, if

$$\left[\!\left[(\bigotimes_{\gamma\in\Gamma}\gamma)\otimes \Box A\right]\!\right] \subseteq [\![B]\!]\,,$$

then

$$\left[\!\left[(\bigotimes_{\gamma\in\Gamma}\gamma)\otimes \Box A\right]\!\right] \subseteq [\![\Box B]\!]\,,$$

or, equivalently, that, if

$$\left[\!\left[\bigotimes_{\gamma\in\Gamma}\gamma\right]\!\right]\circ[\![\Box A]\!]\subseteq[\![B]\!],$$

then

$$\left[\!\left[\bigotimes_{\gamma\in\Gamma}\gamma\right]\!\right]\circ[\![\Box A]\!]\subseteq[\![\Box B]\!].$$

It suffices, then, to prove that, for subsets C, A, and B of M, that if we have

$$C\circ A^{\circ}\subseteq B, \tag{12}$$

then we also have

$$C\circ A^{\circ}\subseteq (B^{\circ})^{\perp\perp}. \tag{13}$$

Now

$$\begin{aligned} C\circ(A^{\circ}) &\subseteq C\circ(A^{\circ})^{\perp\perp}\\ &\subseteq C\circ(A^{\circ\circ})^{\perp\perp}\end{aligned}$$

by Lemma 19

$$\begin{aligned} &\subseteq (C\circ A^{\circ\circ})^{\perp\perp}\\ &\subseteq ((C\circ A^{\circ})^{\circ})^{\perp\perp}\\ &\subseteq (B^{\circ})^{\perp\perp}\end{aligned}$$

which establishes (13).

$\Diamond$**L** Obvious from the definition.

$\Diamond$**R** Likewise obvious.

3.1.2 Completeness

As in Girard [5, 4], we show that there is a phase frame $\mathfrak{M}$ such that $[\![A]\!]=\{\Gamma\mid\Gamma\vdash A\}$. This will establish completeness for the semantics.

DEFINITION 11 The *canonical model*, is given by the following data:

$$\begin{aligned} M &= \{\Gamma\mid\Gamma\text{ a finite multiset of propositions}\}\\ \Gamma\circ\Gamma' &= \Gamma\cup\Gamma'\\ \mathbb{I} &= \emptyset\\ \bot &= \{\Gamma\mid\Gamma\vdash\}\\ X^{\circ} &= \{\Gamma,\Box(A_1\otimes\cdots\otimes A_r)\mid(\Gamma,A_1\otimes\cdots\otimes A_r)\in X\}\\ [\![A]\!] &= \{\Gamma\mid\Gamma\vdash A\}\text{ for }A\text{ atomic.}\end{aligned}$$

Proposition The canonical model is a model.

Proof $\langle M, \circ, \mathbb{I} \rangle$ is clearly a commutative monoid with unit. We note that

$$\Gamma \sqsubseteq \Delta \Leftrightarrow \Gamma \vdash A \rightarrow \Delta \vdash A$$

for all propositions A. So $\perp = \{\Gamma \mid \Gamma \vdash \perp\}$ is clearly downward closed.

To prove that $\perp^\circ \subseteq \perp$, suppose that $\Gamma, A_1, \ldots, A_r \vdash$; then, by cutting with $\Box(A_1 \otimes \cdots A_r) \vdash A_1, \ldots, A_r$, we have $\Gamma, \Box(A_1 \otimes \cdots A_r) \vdash$. (9) is obvious, and (10) takes only a little thought. To see (11), suppose that $m = \Gamma \cup \{\Box(A_1 \otimes \cdots A_r)\} \in A^\circ$; then $m' = \Gamma \cup \{\Box\,\Box\,(A_1 \otimes \cdots A_r)\} \in A^{\circ\circ}$, and, by cutting with $\Box(A_1 \otimes \cdots A_r) \vdash \Box\,\Box\,(A_1 \otimes \cdots A_r)$, we can show that $m \sqsubseteq m'$.

Remark 8 The phase frame of the canonical model is, in fact, relational; the relation is generated by

$$\Gamma, \Box(A_1 \otimes \cdots \otimes A_r) \quad \boldsymbol{\rho} \quad \Gamma, A_1 \otimes \cdots \otimes A_r.$$

Proposition In the canonical model, for any proposition A, $[\![A]\!] = \{\Gamma \mid \Gamma \vdash A\}$.

Proof This is a straightforward induction on the logical complexity of A. We have stipulated the result for A atomic; for composite A, when the principal connective of A is non-modal, we can use the argument in [4, p. 24]. This just leaves two cases:

$A = \Box B$ We can assume inductively that $[\![B]\!] = \{\Gamma \mid \Gamma \vdash B\}$; we first show that $[\![B]\!]^\circ \subseteq \{\Gamma \mid \Gamma \vdash \Box B\}$. Let $m \in [\![B]\!]^\circ$; w.l.o.g we can assume that $m = \Gamma, \Box(P_1 \otimes \cdots \otimes P_r)$ with $\Gamma, P_1, \ldots, P_r \ \vdash \ B$. We have a proof of $\Gamma, \Box(P_1 \otimes \cdots \otimes P_r) \vdash \Box B$ as follows:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma, P_1, \ldots, P_r \ \vdash \ B\end{matrix}}{\Gamma, (P_1 \otimes \cdots \otimes P_r) \ \vdash \ B}}{\Gamma, \Box(P_1 \otimes \cdots \otimes P_r) \ \vdash \ B}\ \Box\mathrm{L}}{\Gamma, \Box(P_1 \otimes \cdots \otimes P_r) \ \vdash \ \Box B}\ \Box\mathrm{R}$$

which shows that $m \in \{\Gamma \mid \Gamma \vdash \Box B\}$.

To prove the converse, we show that $([\![B]\!]^\circ)^\perp \subseteq \{\Gamma \mid \Gamma \vdash B\}^\perp$. Suppose that $\Delta \in ([\![B]\!]^\circ)^\perp$; thus, $\Delta, \Gamma, \Box(P_1 \otimes \cdots \otimes P_r) \vdash$ for all $\Gamma, P_1, \ldots, P_r \vdash B$. In particular, we can take $\Gamma, P_1, \ldots, P_r = B$, and we have $\Delta, \Box B \vdash$. Now suppose we have a Γ such that $\Gamma \vdash \Box B$; by cut, we now have $\Gamma, \Delta \vdash$. This shows that $\Delta \in \{\Gamma \mid \Gamma \vdash B\}^\perp$.

$A = \Diamond B$ Again we assume inductively that $[\![B]\!] = \{\Gamma \mid \Gamma \vdash B\}$; we can also assume (because we have already dealt with the non-modal connectives) that $[\![B^\perp]\!] = \{\Gamma \mid \Gamma \vdash B^\perp\}$. Now suppose that $\Delta \in [\![\Diamond B]\!] = ([\![B^\perp]\!]^\circ)^\perp$. As above, $\Delta, \Box B^\perp \vdash$; cutting with $\vdash \Box B^\perp, \Diamond B$, we have $\Delta \vdash \Diamond B$. Consequently, $[\![\Diamond B]\!] \subseteq \{\Gamma \mid \Gamma \vdash \Diamond B\}$.

Now suppose that $\Gamma \vdash \Diamond B$, and consider an element $(\Delta, \Box(P_1 \otimes \cdots \otimes P_r)) \in [\![B^\perp]\!]^\circ$; that is, $\Delta, P_1, \ldots, P_r \vdash B^\perp$, or, alternatively, $\Delta, P_1, \ldots, P_r, B \vdash$. We now construct a proof of $\Gamma, \Delta, \Box(P_1 \otimes \cdots \otimes P_r) \vdash$ as follows:

$$\dfrac{\dfrac{\dfrac{\vdots}{\dfrac{\Delta, P_1 \otimes \cdots \otimes P_r, B \vdash}{\Delta, \Box(P_1 \otimes \cdots \otimes P_r), B \vdash}}}{\Delta, \Box(P_1 \otimes \cdots \otimes P_r), \Diamond B \vdash}\Diamond\text{L} \qquad \dfrac{\vdots}{\Gamma \vdash \Diamond B}}{\Gamma, \Delta, \Box(P_1 \otimes \cdots \otimes P_r) \vdash}\text{cut}$$

and this shows that $\Gamma \in [\![\Diamond B]\!]$.

Theorem (Completeness for the Phase Semantics) If we have a sequent $\Gamma \vdash \Delta$ which is not valid, then there is some model for which $\left[\!\!\left[\bigotimes_{\gamma \in \Gamma} \Gamma\right]\!\!\right] \not\subseteq \left[\!\!\left[\bigparr_{\delta \in \Delta} \delta\right]\!\!\right]$.

Proof With some rather tedious use of the standard lemmas, we show that it suffices to prove the theorem in the case where the sequent has a single formula on the right and nothing on the left; we therefore have to prove that, if $\vdash A$ is not valid, that there is some model for which $\mathbb{I} \notin [\![A]\!]$. But, by Proposition 5, the standard model has this property.

3.2 Categorial Semantics

We can give a category-theoretic version of the above semantics. Recall that [2, 1]:

Definition 12 A category $\mathfrak{C}$ is $\star$-*autonomous* if

- it is symmetric monoidal closed; that is, there is a monoidal operation $\otimes$ and an internal hom $\multimap$ together with canonical isomorphisms

  $$\mathrm{Hom}(A \otimes B, C) \quad \cong \quad \mathrm{Hom}(A, B \multimap C),$$

  (subject to the usual coherence conditions) and
- there is an object $\perp$ of $\mathfrak{C}$ such that the functor $\cdot \multimap \perp$ is an involution.

We define $(\cdot)^\perp$ to be $\cdot \multimap \perp$, and $\cdot \parr \cdot$ to be $((\cdot)^\perp \otimes (\cdot)^\perp)^\perp$.

(Note that $\star$-autonomous categories are called *linear* categories in [13, 26].)

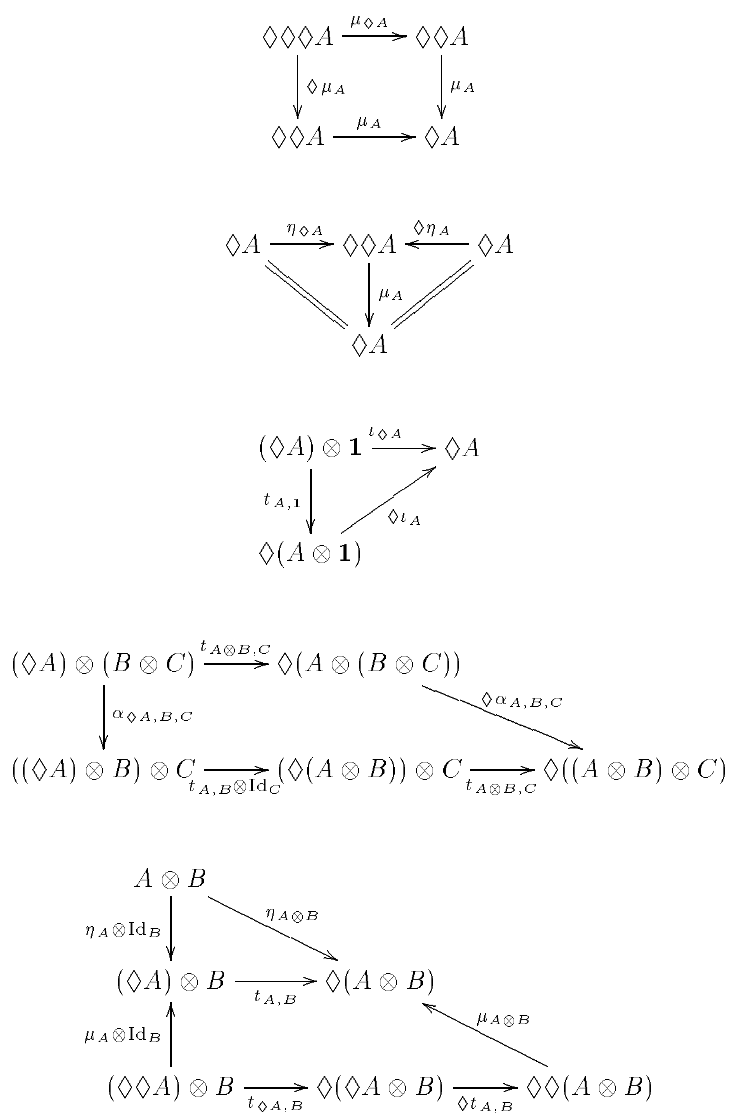
Now a $\star$-autonomous category with (finite and nullary) products and coproducts (one implies the other because of the duality) is a model of classical linear logic; for convenience, we call the product & and the coproduct $\oplus$. We add to this the following:

Definition 13 (Strong Monads) A *strong monad* is a monad $\langle \Diamond(\cdot), \mu, \eta \rangle$ on $\mathfrak{C}$, together with a natural transformation

$$\sigma : \Diamond(\cdot) \otimes \cdot \quad \rightarrow \quad \Diamond(\cdot \otimes \cdot)$$

such that the diagrams in Table 11 commute. (See [16]; cf. [8, 9, 10].)

Table 11 Strong Monads



Here $\Diamond : \mathfrak{C} \to \mathfrak{C}$, $\mu : \Diamond\Diamond \to \Diamond$, $\eta : \mathrm{Id} \to \Diamond$ is a monad, $\sigma : (\Diamond\cdot) \otimes \cdot \to \Diamond(\cdot \otimes \cdot)$ is a strength, and $\alpha : \cdot \otimes (\cdot \otimes \cdot) \to (\cdot \otimes \cdot) \otimes \cdot$ and $\iota : \cdot \otimes \mathbf{1} \to \cdot$ are the associator and unit for the monoidal structure.

Definition 14 (Commutativity for Strong Monads) Given a strong monad on a symmetric monoidal category, let σ' be the natural transformation given by the following composite:

$$A \otimes \Diamond B \to \Diamond B \otimes A \to \Diamond(B \otimes A) \to \Diamond(A \otimes B)$$

A strong monad is *commutative* [20, p. 460] if the following diagram commutes:

$$\begin{array}{ccccc} & \Diamond(\Diamond(A)\otimes B) & \xrightarrow{\Diamond\sigma_{A,B}} & \Diamond\Diamond(A\otimes B) & \\ {\scriptstyle \sigma'_{\Diamond A,B}}\nearrow & & & & \searrow{\scriptstyle \mu} \\ \Diamond A \otimes \Diamond B & & & & \Diamond(A\otimes B) \\ {\scriptstyle \sigma_{A,\Diamond B}}\searrow & & & & \nearrow{\scriptstyle \mu} \\ & \Diamond(A\otimes(\Diamond B)) & \xrightarrow[\Diamond\sigma'_{A,B}]{} & \Diamond\Diamond(A\otimes B) & \end{array}$$

Definition 15 A *linear modal category* is a $\star$-autonomous category together with a commutative strong monad.

We can now define our semantics. First, though, we need a lemma.

Lemma For objects A, B and C of $\mathfrak{C}$, we have natural equivalences

$$\begin{array}{rcl} \mathrm{Hom}(A, B \parr C) & \cong & \mathrm{Hom}(A \otimes B^\perp, C) \\ \mathrm{Hom}(A \otimes B, C) & \cong & \mathrm{Hom}(A, B^\perp \parr C). \end{array}$$

Proof Clear from the definition of $\parr$ and the adjointness between $\otimes$ and $\multimap$.

Definition 16 (Categorial Interpretation: Elementary Definition) Given a $\star$-autonomous category $\mathfrak{C}$ with sums, products, and a commutative strong monad, we associate to each proposition P of our language an object $[\![P]\!]$, and to each proof Π of $\Gamma \vdash \Delta$ a morphism

$$[\![\Pi]\!] : \bigotimes_{\gamma\in\Gamma} [\![\gamma]\!] \ \to \ \bigparr_{\delta\in\Delta} [\![\delta]\!]$$

by induction on the structure of propositions and proofs as follows. (Most of the details for the non-modal case are given in [13], so we will not treat the non-modal connectives in any detail.)

- $\otimes$ We let $[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$. If a proof ends with $\otimes$L, then the last sequent and the penultimate sequent must take values in the same Hom set; so we simply assign the same morphism to the last sequent as was assigned to the penultimate. If a proof ends with $\otimes$R, we can assume inductively that we have, corresponding to the immediate subproofs, morphisms

$$\begin{array}{rcl} [\![\Pi]\!] : \bigotimes_{\gamma\in\Gamma} [\![\gamma]\!] & \to & [\![A]\!] \bigotimes \bigparr_{\delta\in\Delta} [\![\delta]\!] \\ [\![\Pi']\!] : \bigotimes_{\gamma\in\Gamma'} [\![\gamma]\!] & \to & [\![A']\!] \bigotimes \bigparr_{\delta\in\Delta'} [\![\delta]\!] \end{array}$$

We then have (by Lemma 20) morphisms

$$\widetilde{[\![\Pi]\!]} : \bigotimes_{\gamma\in\Gamma} [\![\gamma]\!] \bigotimes \bigotimes_{\delta\in\Delta} [\![\delta]\!]^{\perp} \quad\rightarrow\quad [\![A]\!]$$

$$\widetilde{[\![\Pi']\!]} : \bigotimes_{\gamma\in\Gamma'} [\![\gamma]\!] \bigotimes \bigotimes_{\delta\in\Delta'} [\![\delta]\!]^{\perp} \quad\rightarrow\quad [\![A']\!]$$

We take $\widetilde{[\![\Pi]\!]} \otimes \widetilde{[\![\Pi']\!]}$ and apply Lemma 20 again to get the required morphism

$$\bigotimes_{\gamma\in\Gamma\cup\Gamma'} [\![\gamma]\!] \quad\rightarrow\quad [\![A]\!] \bigotimes [\![A']\!] \bigotimes \bigotimes_{\delta\in\Delta\cup\Delta'} [\![\delta]\!] .$$

$\invamp$ Dual to $\otimes$.

$\multimap$ Very like $\invamp$.

1 We let $[\![\mathbf{1}]\!] = \mathbf{1}$. The left rule for **1** leaves the morphism unchanged; there is only one sequent to which the right rule is applicable (namely $\vdash \mathbf{1}$) and we assign to it the morphism $\mathrm{Id}_{\mathbf{1}}$.

$\perp$ Dual to **1**.

& Similar to $\otimes$: we take $[\![A\&A']\!] = [\![A]\!] \& [\![A']\!]$. To define the morphism for the right rule, we first apply Lemma 20 to obtain morphisms with targets $[\![A]\!]$ and $[\![A']\!]$, combine those morphisms with &, and finally apply Lemma 20. To define the morphism for the left rule, we use a similar process but dual.

$\oplus$ Dual to &.

$\top$ Clear.

$\perp$ Likewise clear.

cut We can assume that the two sequents are of the form $\Gamma \vdash A$ and $A \vdash \Delta$; we simply compose the corresponding morphisms.

$\lozenge$ We define $[\![\lozenge A]\!] = \lozenge [\![A]\!]$. To define a morphism for proofs whose last inference is $\lozenge R$, we can assume, by Lemma 20, that the proof is of the form

$$\Pi : \left\{ \begin{array}{c} \Pi' \\ \vdots \\ \dfrac{\Gamma \vdash A}{\Gamma \vdash \lozenge A} \lozenge\mathrm{R} \end{array} \right.$$

Inductively, we have a morphism $[\![\Pi']\!] : \bigotimes_{\gamma\in\Gamma} [\![\gamma]\!] \rightarrow [\![A]\!]$, and we define a morphism $[\![\Pi]\!] : \bigotimes_{\gamma\in\Gamma} [\![\gamma]\!] \rightarrow [\![\lozenge A]\!]$ by composing with the unit of the monad.

If the last inference of the proof is $\lozenge$L, we can assume that the proof is of the form

$$\Pi : \left\{ \begin{array}{c} \Pi' \\ \vdots \\ \dfrac{\Gamma, A \vdash \lozenge B}{\Gamma, \lozenge A \vdash \lozenge B} \lozenge\mathrm{L} \end{array} \right.$$

Inductively we have a morphism $[\![\Pi']\!]$. We define a morphism for the whole proof by the composite

$$\begin{array}{ccc}
(\Diamond [\![A]\!]) \bigotimes (\bigotimes_{\gamma\in\Gamma} [\![\gamma]\!]) & \overset{[\![\Pi]\!]}{\dashrightarrow} & \Diamond [\![B]\!] \\
\Big\downarrow \sigma_{[\![A]\!], \bigotimes_{\gamma\in\Gamma}[\![\gamma]\!]} & & \Big\uparrow \mu_{[\![B]\!]} \\
 & & \Diamond\Diamond [\![B]\!] \\
 & & \Big\| \\
\Diamond([\![A]\!] \bigotimes (\bigotimes_{\gamma\in\Gamma} [\![\gamma]\!])) & \xrightarrow{\Diamond[\![\Pi']\!]} & \Diamond([\![\Diamond B]\!])
\end{array}$$

$\Box$ Dual to $\Diamond$; we define $[\![\Box A]\!] = (\Diamond [\![A]\!]^{\perp})^{\perp}$.

3.2.1 Functorial Semantics

This is, as it stands, merely a definition, and there is nothing to verify about it. What is not so obvious is that we can make a category in which the objects are linear logic formulae and the morphisms are derivations of the corresponding sequents, and that the above interpretation is a functor. We define the category as follows (see [11, p. 55]):

Definition 17 The *free linear modal category*, $\mathfrak{F}$, has as objects formulae of our language, and an arrow $A \to B$ is a proof of the entailment $A \vdash B$. The $\otimes$ of $\mathfrak{F}$ acts in the usual way on objects, and on morphisms it acts thus:

$$\Pi \otimes \Pi' : \left\{ \dfrac{\dfrac{\begin{array}{c}\Pi\\ \vdots \\ A \vdash B\end{array} \qquad \begin{array}{c}\Pi'\\ \vdots \\ A' \vdash B'\end{array}}{A,\, A' \vdash B \otimes B'}\otimes\mathrm{R}}{A \otimes A' \vdash B \otimes B'}\otimes\mathrm{L} \right.$$

$\multimap$ acts in the obvious way on objects, and on morphisms it acts as

$$\Pi \multimap \Pi' : \left\{ \dfrac{\dfrac{\begin{array}{c}\Pi\\ \vdots \\ A \vdash B\end{array} \qquad \begin{array}{c}\Pi'\\ \vdots \\ A' \vdash B'\end{array}}{B \multimap A',\, A \vdash B'}\multimap L}{B \multimap A' \vdash A \multimap B'}\multimap \mathrm{R} \right.$$

The definitions of $\&$ and $\oplus$ are similar. $\Diamond$ is defined as $\Diamond$ on objects, whereas on arrows it acts as

$$\Diamond\Pi : \left\{ \dfrac{\dfrac{\begin{array}{c}\Pi\\ \vdots \\ A \vdash B\end{array}}{A \vdash \Diamond B}\Diamond\mathrm{R}}{\Diamond A \vdash \Diamond B}\Diamond L \right.$$

η and μ are given by the following arrows:

$$\eta_A : \left\{ \dfrac{\overline{A \vdash A}}{A \vdash \Diamond A}\,\Diamond R \right.$$

$$\mu_A : \left\{ \dfrac{\dfrac{\dfrac{\overline{A \vdash A}}{A \vdash \Diamond A}\,\Diamond \mathrm{R}}{\Diamond A \vdash \Diamond A}\,\Diamond \mathrm{L}}{\Diamond\Diamond A \vdash \Diamond A}\,\Diamond L \right.$$

Equality between arrows will be defined below.

If we want a category here (in which, for example, the associativity of $\otimes$ holds up to equality) we cannot simply have the trivial equality of proofs here. The tricky points are naturality and coherence. Naturality means, for example, that, if we have objects A, A', B, B', C and C' of our category, and if we have morphisms $\phi : A \to A'$, $\psi : B \to B'$, and $\chi : C \to C'$, then the following diagram commutes:

$$\begin{array}{ccc} A \otimes (B \otimes C) & \longrightarrow & (A \otimes B) \otimes C \\ {\scriptstyle \phi\otimes(\psi\otimes\chi)}\downarrow & & \downarrow{\scriptstyle (\phi\otimes\psi)\otimes\chi} \\ A' \otimes (B' \otimes C') & \longrightarrow & (A' \otimes B') \otimes C' \end{array}$$

In our case, of course, the objects are formulae and the morphisms are derivations; we assume that we have derivations Φ of $A \vdash A'$, Ψ of $B \vdash B'$, and X of $C \vdash C'$. Now the top right path around the diagram leads to this proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \vdash A} \quad \overline{B \vdash B}}{A, B \vdash A \otimes B} \quad \overline{C \vdash C}}{A, B, C \vdash (A \otimes B) \otimes C}}{A, B \otimes C \vdash (A \otimes B) \otimes C}}{A \otimes (B \otimes C) \vdash (A \otimes B) \otimes C} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\Phi \\ \vdots \\ A \vdash A'\end{matrix} \quad \begin{matrix}\Psi \\ \vdots \\ B \vdash B'\end{matrix}}{A, B \vdash A' \otimes B'} \quad \begin{matrix}\mathsf{X} \\ \vdots \\ C \vdash C'\end{matrix}}{A, B, C \vdash (A' \otimes B') \otimes C'}}{A \otimes B, C \vdash (A' \otimes B') \otimes C'}}{(A \otimes B) \otimes C \vdash (A' \otimes B') \otimes C'}}{A \otimes (B \otimes C) \vdash (A' \otimes B') \otimes C'}\,\mathrm{cut}$$

whereas the bottom left route leads to the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\Phi \\ \vdots \\ A \vdash A'\end{matrix} \quad \dfrac{\begin{matrix}\Psi \\ \vdots \\ B \vdash B'\end{matrix} \quad \begin{matrix}\mathsf{X} \\ \vdots \\ C \vdash C'\end{matrix}}{B, C \vdash B' \otimes C'}}{A, B, C \vdash A' \otimes (B' \otimes C')}}{A, B \otimes C \vdash A' \otimes (B' \otimes C')}}{A \otimes (B \otimes C) \vdash A' \otimes (B' \otimes C')} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{A' \vdash A'} \quad \overline{B' \vdash B'}}{A', B' \vdash A' \otimes B'} \quad \overline{C' \vdash C'}}{A', B', C' \vdash (A' \otimes B') \otimes C'}}{A', B' \otimes C' \vdash (A' \otimes B') \otimes C'}}{A' \otimes (B' \otimes C') \vdash (A' \otimes B') \otimes C'}}{A \otimes (B \otimes C) \vdash (A' \otimes B') \otimes C'}\,\mathrm{cut}$$

These are clearly different proofs. Thus, in order to regard linear logic proofs as the morphisms of a category, we are led to impose (implicitly or explicitly) a non-trivial equality between proofs; as Troelstra remarks [26, p. 91] a good deal of freedom as to how we do this.

We are, then, here following [13] for the treatment of the non-modal fragment of our logic; our task will be to extend this to the modal operators. The additional equations to be satisfied are those given by the diagrams in Table 11. The top diagram, for example, expresses the equality $\mu_A \circ \mu_{\Diamond A} = \mu_A \circ \Diamond \mu_A$; now the first is given by the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Diamond A \vdash \Diamond A}\,\mathrm{Ax}_{\Diamond A}}{\Diamond A \vdash \Diamond\Diamond A}\,\Diamond\mathrm{R}}{\Diamond\Diamond A \vdash \Diamond\Diamond A}\,\Diamond\mathrm{L}}{\Diamond\Diamond\Diamond A \vdash \Diamond\Diamond A}\,\Diamond\mathrm{L} \qquad \begin{array}{c}\mu_A\\ \vdots\\ \Diamond\Diamond A \vdash \Diamond A\end{array}}{\Diamond\Diamond\Diamond A \vdash \Diamond A}\,\mathrm{cut}$$

whereas the second is given by the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathrm{Ax}_{A}}{A \vdash \Diamond A}\,\Diamond\mathrm{R}}{\Diamond A \vdash \Diamond A}\,\Diamond\mathrm{L}}{\Diamond\Diamond A \vdash \Diamond A}\,\Diamond\mathrm{L}}{\Diamond\Diamond A \vdash \Diamond\Diamond A}\,\Diamond\mathrm{R}}{\Diamond\Diamond\Diamond A \vdash \Diamond\Diamond A}\,\Diamond\mathrm{L} \qquad \begin{array}{c}\mu_A\\ \vdots\\ \Diamond\Diamond A \vdash \Diamond A\end{array}}{\Diamond\Diamond\Diamond A \vdash \Diamond A}\,\mathrm{cut}$$

We want these two proofs to be equal. We need similar equalities for the other diagrams in Table 11, and for the other diagrams defining a $\star$-autonomous category.

With this definition of equality between proofs, then, we have defined a category, $\mathfrak{F}$, in which the objects are linear logic formulae and in which the morphisms are proofs of entailments. Let us call this category $\mathfrak{T}$; it is something very like a term model. We now have

Definition 18 (Categorial Interpretation: Functorial Definition) Given a $\star$-autonomous category $\mathfrak{C}$, together with a monoid with strength, an interpretation of our logic in $\mathfrak{C}$ is a monoidal functor $\mathfrak{F} \to \mathfrak{C}$ which preserves the dualising object, the monoids, and the strengths.

Given this, it is reasonably easy (but tediously bureaucratic) to carry through the programme of Martí-Oliet and Meseguer, and prove soundness and completeness for this semantics. $\mathfrak{F}$ corresponds to the $\mathcal{L}[T]$ of [13, Definition 41], and we first prove the equivalent of their Theorem 43, namely that the elementary and functorial definitions of our categorical semantics coincide. Given an assignment of objects of some linear modal category $\mathfrak{C}$ to atomic formulae of our language, it is easy enough to unwind the definitions of the objects and morphisms of $\mathfrak{F}$ and thereby define a functor $\mathfrak{F} \to \mathfrak{C}$. We can then find, for each proof of $A \vdash B$, an arrow of $\mathfrak{F}(A, B)$ which maps onto the arrow of $\mathfrak{C}([\![A]\!], [\![B]\!])$ corresponding to the proof. On the other hand, given a functor $\mathfrak{F} \to \mathfrak{C}$, we can obtain from it an assignment of objects of $\mathfrak{C}$ to atomic formulae of our language, and thus to find an elementary semantic assignment for our language.

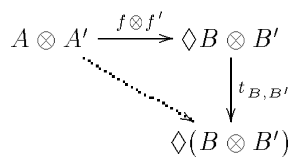
Example 2 (Phase Frames as Categories) Given a phase frame $\mathfrak{M}$, we can regard its set of facts as a category $\mathfrak{C}_{\mathfrak{M}}$ (the morphisms are inclusions), and an interpretation of our logic in $\mathfrak{M}$ can also be regarded as a categorial interpretation of $\mathfrak{F} \to \mathfrak{C}_{\mathfrak{M}}$. In particular, the canonical phase frame is the extensional collapse of $\mathfrak{T}$.

3.2.2 The Kleisli Category

Let $\mathfrak{C}$ be a linear modal category, and consider now its Kleisli category $\mathfrak{C}_{\Diamond}$; objects are the same as objects of $\mathfrak{C}$, whereas morphisms in $\mathfrak{C}_{\Diamond}(A, B)$ are the same as morphisms in $\mathfrak{C}(A, \Diamond B)$. We have

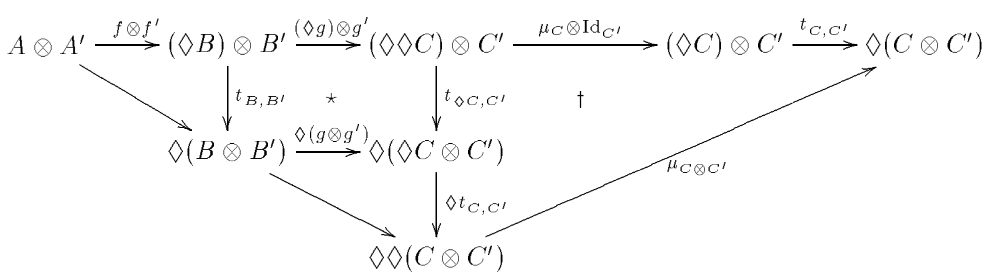
Proposition $\otimes$ can be extended to a functor $\mathfrak{C}_{\Diamond} \times \mathfrak{C} \to \mathfrak{C}_{\Diamond}$.

Proof We define $\underline{f} \otimes f'$ for $\underline{f} \in \mathfrak{C}_{\Diamond}(A, B)$ and $f' \in \mathfrak{C}(A', B')$ by the composite



where $\underline{f}$ corresponds to $f : A \to \Diamond B$ in $\mathfrak{C}$ and where the tensor products in the above diagram are those of $\mathfrak{C}$.

We have to show that this is a functor. Firstly, we show that it respects composition, namely that, if we have $\underline{f} : A \to B$, $\underline{g} : B \to C$ in $\mathfrak{C}_{\Diamond}$, and $f' : A' \to B'$ and $g' : B' \to C'$ in $\mathfrak{C}$, then $(\underline{g} \circ \underline{f}) \otimes (g' \circ f') = (\underline{g} \otimes g') \circ (\underline{f} \otimes f')$ (in $\mathfrak{C}_{\Diamond}$). Now suppose that $\underline{f}$ corresponds to $f : A \to \Diamond B$, and $\underline{g}$ corresponds to $f : B \to \Diamond C$ (in $\mathfrak{C}$), and consider the following diagram.



The top edge is the $\mathfrak{C}$-morphism corresponding to $(\underline{g} \circ \underline{f}) \otimes (g' \circ f')$, whereas the composite of the bottom two edges is the $\mathfrak{C}$-morphism corresponding to $(\underline{g} \otimes g') \circ (\underline{f} \otimes f')$. However, $\star$ commutes because t is a natural transformation, and $\dagger$ commutes because of the bottom diagram in Table 11. (The small triangles commute by definition.)

Finally we show that $\otimes$ respects identities. For an object A, its identity $\underline{\mathrm{Id}}_A$ in $\mathfrak{C}_{\Diamond}$ corresponds to η_A in $\mathfrak{C}$; so, to show that $\underline{\mathrm{Id}}_A \otimes \mathrm{Id}_{A'} = \underline{\mathrm{Id}}_{A \otimes A'}$, we have to show that $t_{A,A'} \circ (\eta_A \otimes \mathrm{Id}) = \eta_{A \otimes A'}$; but this follows from the bottom diagram in Table 11.

4 Interpretations

We will outline here the notion of an interpretation of one language in another, where the languages are extensions of linear logic by some or all of our sets of rules.

Definition 19 (Simulating Interpretations) Let $\mathcal{L}_1$ and $\mathcal{L}_1$ be two languages in classical linear logic, together with strong modalities and axioms corresponding to rewrites $\leadsto_1$ and $\leadsto_2$, respectively. A *simulating interpretation* of $\langle \mathcal{L}_1, \leadsto_1 \rangle$ in $\langle \mathcal{L}_2, \leadsto_2 \rangle$ is a mapping assigning, to each atom a of $\mathcal{L}_1$, a state proposition $\overline{a}$ of $\mathcal{L}_2$, such that, if we extend this mapping in the obvious way to state propositions of $\mathcal{L}_1$, then, if we have $a \leadsto_1 b$, we also have $\overline{a} \leadsto_1 \overline{b}$.

We now have

Proposition (Simulating Interpretations and Logical Consequence) If we have a simulating interpretation of $\mathcal{L}_1$ in $\mathcal{L}_2$, then, if $\Gamma \vdash \Delta$ is a valid entailment in $\mathcal{L}_1$, then $\overline{\Gamma} \vdash \overline{\Delta}$ is a valid entailment in $\mathcal{L}_2$.

Proof An obvious induction.

Definition 20 (Bisimulating Interpretations) Let $\mathcal{L}_1$ and $\mathcal{L}_1$ be two languages in classical linear logic, together with strong modalities, axioms corresponding to rewrites $\leadsto_1$ and $\leadsto_2$, and operators $\boldsymbol{\sigma}_1(\cdot)$, $\boldsymbol{\tau}_1(\cdot)$, and $\boldsymbol{\sigma}_2(\cdot)$, $\boldsymbol{\tau}_2(\cdot)$, respectively. A *bisimulating interpretation* of $\langle \mathcal{L}_1, \leadsto_1 \rangle$ in $\langle \mathcal{L}_2, \leadsto_2 \rangle$ is a mapping assigning, to each atom a of $\mathcal{L}_1$, a state proposition $\overline{a}$ of $\mathcal{L}_2$, such that, if we extend this mapping in the obvious way to state propositions of $\mathcal{L}_1$, then, if we have $a \leadsto_1 b$, we also have $\overline{a} \leadsto_1 \overline{b}$ and also that, if we have $x \leadsto_2 y$, we have state propositions a and b of $\mathcal{L}_1$ with $x \dashv\vdash_2 \overline{x}$, $y \dashv\vdash_2 \overline{b}$, and $a \leadsto_1 b$.

Proposition (Bisimulating Interpretations and Logical Consequence) If we have a bisimulating interpretation of $\mathcal{L}_1$ in $\mathcal{L}_2$, then, if $\Gamma \vdash \Delta$ is a valid entailment in $\mathcal{L}_1$, then $\overline{\Gamma} \vdash \overline{\Delta}$ is a valid entailment in $\mathcal{L}_2$.

Proof The obvious induction.

Example 3 (Balls and String: Interpretations) Consider Example 1; let $\mathcal{L}_1$ be the language in which this example is formulated. This has primitives $\mathrm{at}(b_i, l_j)$, $\mathrm{slack}(s)$, and $\mathrm{taut}(s)$. Let $\mathcal{L}_1$ be the language with primitives $\mathrm{above}(b_i, l_j)$, $\mathrm{below}(b_i, l_j)$, $\mathrm{slack}(s)$, and $\mathrm{taut}(s)$. We can translate from $\mathcal{L}_1$ to $\mathcal{L}_2$ by

$$
\begin{array}{rcl}
\mathrm{at}(b_i, l_j) & \mapsto & \mathrm{above}(b_i, l_{j-1}) \otimes \mathrm{below}(b_i, l_{j+1}) \\
\mathrm{slack}(s) & \mapsto & \mathrm{slack}(s) \\
\mathrm{taut}(s) & \mapsto & \mathrm{taut}(s)
\end{array}
$$

and in the other direction by

$$
\begin{array}{rcl}
\mathrm{above}(b_i, l_j) & \mapsto & \mathrm{at}(b_i, l_{j+1}) \oplus \mathrm{at}(b_i, l_{j+2}) \oplus \ldots \\
\mathrm{below}(b_i, l_j) & \mapsto & \mathrm{at}(b_i, l_{j-1}) \oplus \mathrm{at}(b_i, l_{j-2}) \oplus \ldots \\
\mathrm{slack}(s) & \mapsto & \mathrm{slack}(s) \\
\mathrm{taut}(s) & \mapsto & \mathrm{taut}(s)
\end{array}
$$

These mappings clearly set up an isomorphism between the state propositions of $\mathcal{L}_1$ and $\mathcal{L}_2$, and so we can, by transport of structure, define a rewrite relation on $\mathcal{L}_2$ which bisimulates that on $\mathcal{L}_1$. By Proposition 8, we can just as well look for proofs of the relevant sequents in $\mathcal{L}_2$ as in $\mathcal{L}_1$.

A The Connection with Golog

A.1 Modalities

A.1.1 Definitions

In this appendix we show how these modalities can be connected with previous work on GOLOG in [32]. In particular, we should recall that, for actions α and β, we can define their sequential composition, $\alpha\,;\beta$, by

$$\alpha\,;\beta \quad \stackrel{\text{df}}{=} \quad \forall X.\,(\alpha \otimes \boldsymbol{\sigma}(X)^{\perp}) \parr (\boldsymbol{\sigma}(X) \otimes \beta) \tag{14}$$

We can now define

DEFINITION 21 A GOLOG *class of actions* $\mathcal{G}$ is a set of actions such that

- $\mathbf{1} \in \mathcal{G}$, and
- If α, $\beta \in \mathcal{G}$, then $\alpha\,;\beta \in \mathcal{G}$.

We now define modal operators as follows.

DEFINITION 22 We fix, for the moment, a suitable class $\mathcal{G}$ of actions; quantified Greek letters will range over this class.

$$\begin{aligned} \Diamond A &\stackrel{\text{df}}{=} \exists \alpha.\,\alpha \multimap A \\ \Box A &\stackrel{\text{df}}{=} \forall \alpha.\,\alpha \otimes A \end{aligned}$$

We will prove that the two modalities – $\Diamond$ and $\Box$ – are strong modal operators: their sequent calculus rules are given in Table 2.

A.1.2 Rules for Modalities

Now for the proof that operators thus defined are modalities. We want, then, to prove that the rules in Table 2 are admissible; however, we notice that these rules are equivalent to another set – namely, those in Table 3.

Now we want to show that these rules are admissible. Notice, however, that they are equivalent to another set, which are the ones we shall actually prove.

PROPOSITION The rules of $\mathsf{LL}_{\Diamond}$, given in Table 2 are equivalent to the rules of $\mathsf{LL}_{\Diamond}{}'$, given in Table 3.

PROOF Suppose first that we have an operator satisfying $\mathsf{LL}_{\Diamond}$. The first pair of $\mathsf{LL}_{\Diamond}{}'$ are special cases of the first pair of $\mathsf{LL}_{\Diamond}$. We prove the second pair as follows (see [28, Definition 9.1.3]):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A,\, A^{\perp} \vdash}}{\Box A,\, A^{\perp} \vdash}}{\Box A,\, \Diamond A^{\perp} \vdash}}{\Box A \vdash (\Diamond A^{\perp})^{\perp}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\vdash A,\, A^{\perp}}}{\vdash A,\, \Diamond A^{\perp}}}{\vdash \Box A,\, \Diamond A^{\perp}}}{(\Diamond A^{\perp})^{\perp} \vdash \Box A}$$

The rules of $\mathsf{LL}_{\Diamond}{}'$ are thus admissible, given the rules of $\mathsf{LL}_{\Diamond}$.

Conversely, suppose that we have the rules of $\mathsf{LL}_{\Diamond}{}'$ (Table 3). We prove the admissibility of $\mathsf{LL}_{\Diamond}$ (Table 2) as follows.

$\Box$L Note that we can prove $\Box A \vdash A$ as follows:

$$\dfrac{\dfrac{}{\Box A \vdash (\Diamond A^{\perp})^{\perp}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\vdash A^{\perp},\, A}}{\vdash \Diamond A^{\perp},\, A}}{(\Diamond A^{\perp})^{\perp} \vdash A}}{\Box A \vdash A}\ \text{cut}$$

Suppose now that we have a proof of $\Gamma,\, A \vdash \Delta$; we simply cut it with the above proof of $\Box A \vdash A$, and we have a proof of $\Gamma,\, \Box A \vdash \Delta$.

$\Box$R$_1$ Suppose that we have a proof Π of $\Gamma,\, \Box A \vdash B, \Delta$. From it we can, by a cut with $(\Diamond A^{\perp})^{\perp} \vdash \Box A$, obtain a proof Π' of $\Gamma,\, (\Diamond A^{\perp})^{\perp} \vdash B,\, \Delta$, and from this, by the involutivity of negation and the rules for $^{\perp}$, we can obtain a proof Π'' of $B^{\perp} \vdash \Diamond A^{\perp},\, \Delta,\, \Gamma^{\perp}$.

We now construct a proof of $\Gamma,\, \Box A \vdash \Box B,\, \Delta$ as follows:

$$\dfrac{\dfrac{\begin{array}{c}\Pi'' \\ \vdots \\ \dfrac{B^{\perp} \vdash \Diamond(A^{\perp}),\, \Delta,\, \Gamma^{\perp}}{\Diamond(B^{\perp}) \vdash \Diamond(A^{\perp}),\, \Delta,\, \Gamma^{\perp}}\ \dagger \\ \vdots\ * \\ \Gamma,\, \Box A,\, \Diamond B^{\perp} \vdash \Delta\end{array}}{\Gamma,\, \Box A \vdash (\Diamond B^{\perp})^{\perp},\, \Delta} \qquad \dfrac{}{(\Diamond B^{\perp})^{\perp} \vdash \Box B}}{\Gamma,\, \Box A \vdash \Box B,\, \Delta}\ \text{cut}$$

where $*$ is a sequence of cuts with $(\Diamond \gamma^{\perp})^{\perp} \vdash \Box\gamma$ followed by a sequence of applications of the involutivity of negation and $^{\perp}$ rules, and $\dagger$ is an application of the left $\Diamond$-rule from Table 3.

$\Diamond$R$_2$ Similar to $\Box$R$_1$.

$\Diamond$L$_1$ Similar to $\Box$R$_1$.

$\Diamond\mathbf{L}_2$ Similar to $\Box\mathrm{R}_1$.

$\Diamond\mathbf{R}$ Similar to $\Box$L.

A.1.3 Admissibility of the Modal Operators

We now take the operators defined in Section A.1.1 and show that they are modalities, i.e. that the sequent calculus rules are admissible for them. By the results of the previous section, it suffices to prove the admissibility of the rules of $\mathsf{LL}_\Diamond{}'$; this we now do.

PROPOSITION If A is any proposition, the sequent

$$\Box A \vdash (\Diamond A^\perp)^\perp$$

is valid.

PROOF

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha\vdash\alpha} \qquad \dfrac{\dfrac{}{A\vdash A}}{A,\,A^\perp\vdash}}{\alpha,\,A,\,\alpha\multimap A^\perp\vdash}}{\alpha\otimes A,\,\alpha\multimap A^\perp\vdash}}{\forall\beta.\,\beta\otimes A,\,\alpha\multimap A^\perp\vdash}}{\Box A,\,\alpha\multimap A^\perp\vdash}}{\Box A,\exists\alpha.\,\alpha\multimap A^\perp\vdash}\;*}{\Box A,\,\Diamond A^\perp\vdash}}{\Box A\vdash(\Diamond A^\perp)^\perp}
$$

where a new action variable α is introduced at $*$.

PROPOSITION If A is any proposition, the sequent

$$(\Diamond A^\perp)^\perp \vdash \Box A$$

is valid.

PROOF

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha\vdash\alpha} \qquad \dfrac{\dfrac{}{A\vdash A}}{\vdash A^\perp,\,A}}{\alpha\vdash A^\perp,\,\alpha\otimes A}}{\vdash \alpha\multimap A^\perp,\,\alpha\otimes A}}{\vdash \exists\beta.\,\beta\multimap A^\perp,\,\alpha\otimes A}}{\vdash \Diamond A^\perp,\,\alpha\otimes A}}{\vdash \Diamond A^\perp,\,\forall\alpha.\,\alpha\otimes A}\;*}{\vdash \Diamond A^\perp,\,\Box A}}{(\Diamond A^\perp)^\perp\vdash\Box A}
$$

where α is a new propositional variable introduced at $*$.

PROPOSITION If $\Gamma \vdash A, \Delta$ is provable, then so is $\Gamma \vdash \Diamond A, \Delta$.

PROOF Let Π be a proof of $\Gamma \vdash A, \Delta$; we construct a proof of $\Gamma \vdash \Diamond A, \Delta$ as follows.

$$\begin{array}{c} \Pi \\ \vdots \\ \Gamma \vdash A, \Delta \\ \hline \Gamma, \mathbf{1} \vdash A, \Delta \\ \hline \Gamma \vdash \mathbf{1} \multimap A, \Delta \\ \hline \Gamma \vdash \exists \alpha.\, \alpha \multimap A, \Delta \\ \hline \Gamma \vdash \Diamond A, \Delta \end{array}$$

LEMMA For any X, β_i, B, B_i,

$$X \multimap ((\beta_1 \multimap B) \parr B_1 \parr \ldots) \dashv\vdash (X \otimes \beta) \multimap (B_1 \parr B_2 \parr \ldots)$$

is valid.

PROOF A routine calculation with the definition of $\multimap$ and the De Morgan equivalences:

$$\begin{array}{rl} & X \multimap ((\beta \multimap B) \parr B_1 \parr \ldots) \\ \cong & X^\perp \parr ((\beta \multimap B) \parr B_1 \parr \ldots) \\ \cong & X^\perp \parr ((\beta^\perp \parr B) \parr B_1 \parr \ldots) \\ \cong & (X^\perp \parr \beta^\perp) \parr (B \parr B_1 \parr \ldots) \\ \cong & (X \otimes \beta)^\perp \parr (B \parr B_1 \parr \ldots) \\ \cong & (X \otimes \beta) \multimap (B \parr B_1 \parr \ldots) \end{array}$$

LEMMA For any situation X, and for propositions α and β,

$$\alpha \multimap X, \alpha\, ; \beta \vdash X \otimes \beta$$

is valid.

PROOF

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha \vdash \alpha} \qquad \dfrac{\dfrac{}{X \vdash X}}{X, X^\perp \vdash}}{\alpha \multimap X, \alpha, X^\perp \vdash}}{\alpha \multimap X, \alpha \otimes X^\perp \vdash} \qquad \dfrac{}{X \otimes \beta \vdash X \otimes \beta}}{\dfrac{\alpha \multimap X, (\alpha \otimes X^\perp) \parr (X \otimes \beta) \vdash X \otimes \beta}{\alpha \multimap X, \forall Y. ((\alpha \otimes Y^\perp) \parr (Y \otimes \beta)) \vdash X \otimes \beta}}}{\alpha \multimap X, \alpha\, ; \beta \vdash X \otimes \beta}$$

LEMMA For any action α and any situation X, and for propositions $B, B_1, B_2, \ldots,$

$$(\alpha \multimap X) \otimes (X \multimap (\Diamond B \parr B_1 \parr \ldots)) \vdash \Diamond B \parr B_1 \parr \ldots$$

is valid.

PROOF

$$\dfrac{\dfrac{\vdots}{B \parr B_1 \parr \ldots \vdash B,\, B_1,\, \ldots} \qquad \dfrac{\dfrac{\vdots}{X \otimes \beta \vdash X \otimes \beta}}{\alpha \multimap X, (\alpha\,;\beta) \vdash X \otimes \beta}\,\P}{\dfrac{(X \otimes \beta) \multimap (B \parr B_1 \parr \ldots),\, (\alpha\,;\beta) \vdash B,\, B_1,\, \ldots}{\dfrac{\alpha \multimap X,\, (X \otimes \beta) \multimap (B \parr B_1 \parr \ldots) \vdash (\alpha\,;\beta \multimap B,\, B_1,\, \ldots}{\dfrac{\alpha \multimap X,\, (X \otimes \beta) \multimap (B \parr B_1 \parr \ldots) \vdash \Diamond B,\, B_1,\, \ldots}{\dfrac{\alpha \multimap X,\, X \multimap ((\beta \multimap B) \parr B_1 \parr \ldots) \vdash \Diamond B,\, B_1,\, \ldots}{\dfrac{\alpha \multimap X,\, X \multimap (\Diamond B \parr B_1 \parr \ldots) \vdash \Diamond B \parr B_1 \parr \ldots}{(\alpha \multimap X) \otimes (X \multimap (\Diamond B \parr B_1 \parr \ldots)) \vdash \Diamond B \parr B_1 \parr \ldots}\,*}\,\dagger}\,\ddagger}}}$$

At $*$ we use the definition of $\Diamond$, followed by the left rule for $\exists$; we therefore have to introduce a new action variable β. $\dagger$ is a cut with Lemma 21. At $\ddagger$ we use the definition of $\Diamond$, followed by the right rule for $\exists$. We can thus substitute whichever terms we like for the bound variable, and we make the choice shown; we have here got to use the assumption that our actions form a GOLOG-class, so $\alpha\,;\beta$ is an action, and thus a suitable term to substitute for an action variable. $\P$ is a cut with Lemma 22.

PROPOSITION If $A \vdash \Diamond B,\, \Delta$ is provable, then so is $\Diamond A \vdash \Diamond B,\, \Delta$.

PROOF Suppose that we have a proof Π of $A \vdash \Diamond B,\, \Delta$. Let $\Delta = B_1, B_2, \ldots$. We construct a proof of $\Diamond A \vdash \Diamond B,\, \Delta$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi}{\vdots}}{A \vdash \Diamond B,\, B_1,\, \ldots}}{\dfrac{A \vdash \Diamond B \parr B_1 \parr \ldots}{\vdash A \multimap (\Diamond B \parr B_1 \parr \ldots)}} \qquad \dfrac{\text{Lemma 23}}{\vdots \atop \alpha \multimap A,\, A \multimap (\Diamond B \parr \Diamond B_1 \parr \ldots) \vdash \Diamond B,\, B_1,\, \ldots}}{\dfrac{\alpha \multimap A \vdash \Diamond B,\, B_1,\, \ldots}{\dfrac{\exists \alpha.\, \alpha \multimap A \vdash \Diamond B,\, B_1,\, \ldots}{\Diamond A \vdash \Diamond B,\, B_1,\, \ldots}}}\,\text{cut}$$

THEOREM (MODALITIES VIA GOLOG) The operators $\Box$ and $\Diamond$ defined in Definition 22 are modalities of $\mathsf{LL}_\Diamond$.

PROOF Propositions 10, 11, 12, and 13.

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